

INVERSION FORMULAS AND RANGE CHARACTERIZATIONS FOR THE ATTENUATED GEODESIC RAY TRANSFORM

YERNAT M. ASSYLBEKOV, FRANÇOIS MONARD, AND GUNTHER UHLMANN

ABSTRACT. We present two range characterizations for the attenuated geodesic X-ray transform defined on pairs of functions and one-forms on simple surfaces. Such characterizations are based on first isolating the range over sums of functions and one-forms, then separating each sub-range in two ways, first by implicit conditions, second by deriving new inversion formulas for sums of functions and one-forms.

1. INTRODUCTION

Let (M, g) be a smooth compact oriented Riemannian surface with boundary ∂M , with unit tangent bundle $SM := \{(x, v) \in TM : |v|_{g(x)} = 1\}$ and inward/outward boundaries

$$\partial_{\pm} SM = \{(x, v) \in SM : x \in \partial M, \pm \langle v, \nu_x \rangle_{g(x)} \geq 0\},$$

where ν_x is the unit inward normal at $x \in \partial M$. Denote $\varphi_t : SM \rightarrow SM$ the geodesic flow, written as $\varphi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$ and defined for $-\tau(x, -v) \leq t \leq \tau(x, v)$, where $\tau(x, v)$ is the first exit time of the geodesic starting at (x, v) . Throughout the paper, we assume that (M, g) is *simple*, meaning that the boundary is strictly convex and that any two points on the boundary are joined by a unique minimizing geodesic. In particular, this implies that (M, g) is simply connected and that $\tau(x, v)$ is bounded on SM (i.e., (M, g) is non-trapping). For $a \in C^\infty(M, \mathbb{C})$, the object of study is the *attenuated geodesic ray transform* $I_a : C^\infty(SM) \rightarrow C^\infty(\partial_+ SM)$ defined for $f \in C^\infty(SM)$ as

$$I_a f(x, v) = \int_0^{\tau(x, v)} f(\varphi_t(x, v)) \exp \left(\int_0^t a(\gamma_{x,v}(s)) ds \right) dt, \quad (x, v) \in \partial_+ SM. \quad (1)$$

The present article aims at providing range characterizations for this transform over pairs of functions and one-forms, or equivalently, when the integrand f above takes the form $f(x, v) = f_0(x) + \alpha_x(v)$ for $[f_0, \alpha]$ a pair of a function and a one-form. As the transform above models some medical imaging modalities such as Computerized Tomography and Ultrasound Doppler Tomography in media with variable refractive index, range characterizations are useful to project noisy data onto the range of a given measurement operator before inverting for the unknown (f_0 or α here). In media with constant refractive index, modelled by the Euclidean metric in the parallel geometry, the problem was extensively studied [19, 17, 2, 8, 38], and the range characterization was already a challenging issue yet to be solved [18].

Recently, range characterizations for the attenuated transform on convex Euclidean domains were provided in terms of Hilbert transforms with respect to A-analytic function theory *à la* Bukhgeim, treating the case of functions [31], vector fields [30] and two-tensors [29], though such results are limited to Euclidean settings as A-analytic function theory has not yet been developed on general surfaces.

In the case of manifolds with no symmetries, parallel geometry does not exist and one must work with fan-beam coordinates. The scalar case has been studied in [32, 15] in the geodesic case, and in [14] in the Euclidean, fan-beam case, mainly focused on injectivity, stability and inversion procedures.

On to range characterizations, the first one in terms of boundary operators was provided by Pestov and Uhlmann in [26], later generalized to the case of transport with unitary connection, with further applications to the range characterization of the unattenuated transform over higher-order tensors [22]. Recently in [16], the range characterization in [26] was proved by the second author to be a generalization of the classical moment conditions in the Euclidean setting.

In the approach coming from [26], there is a boundary operator P which only depends on the scattering relation and the fiberwise Hilbert transform, and which characterizes the unattenuated transform over functions and one-forms. Further splitting of P into the sum $P_+ + P_-$ allows to separate ranges over functions and one-forms. A major challenge in the attenuated case is that, despite the fact that a similar operator exists for the ray transform over pairs (a fact which is one of the first features of this article), the splitting mentioned above is no longer straightforward. We then propose two approaches to separate the sub-ranges within the range over pairs.

The first approach is an implicit description given by adding constraints on the preimage by the P operator above, while the second one relies on inversion formulas for each term of the pair, from the data of both.

In [15], the second author provides inversion formulas for the attenuated ray transforms for functions and vector fields, including one which takes the form of a Fredholm equation, in which the operator may depend on the attenuation coefficient. The formulas presented here allows inversions for pairs (function + one-form) modulo natural obstructions. Moreover, the integrands can be supported up to the boundary. The formulas are exact provided that one can invert the unattenuated transform over functions and solenoidal vector fields. In that regard, the approach does not suffer from whether the attenuation is too low or too high as in [15]. Additionally, it is generalized to complex-valued attenuations, which requires using both holomorphic and antiholomorphic integrating factors, as in the first inversion procedure presented in [32]. An additional tool which is introduced and allows to extract information in a systematic fashion, is a way to turn transport solutions with holomorphic right-hand sides into holomorphic solutions themselves, by manipulating their boundary values. In some sense, this operation is to be understood as a change in the qualitative features of the solutions by “data” processing.

We now state the main results and give an outline of the remainder of the article in the next section.

2. STATEMENTS OF MAIN RESULTS

In what follows, for F some function space (C^k , L^p , H^k , etc.), we denote by $\mathcal{F}(M, \mathbb{C})$ the corresponding space of pairs $[\alpha, f]$ with α a 1-form and f a function on M . In particular, $\mathcal{C}^\infty(M, \mathbb{C})$ is the space of pairs $[\alpha, f]$, with $\alpha \in C^\infty(\Lambda^1(M), \mathbb{C})$ and $f \in C^\infty(M, \mathbb{C})$. Then \mathcal{I}_a denotes the restriction of I_a to $\mathcal{C}^\infty(M, \mathbb{C})$:

$$\mathcal{I}_a[\alpha, f](x, v) := I_a^1 \alpha(x, v) + I_a^0 f(x, v), \quad (x, v) \in \partial_+ SM,$$

where I_a^1 and I_a^0 are the restrictions of I_a to 1-forms and functions on M , respectively.

Let $X(x, v) = \frac{d}{dt}|_{t=0} \varphi_t(x, v)$ denote the generator of the geodesic flow of g , a global section of $T(SM)$. Here and below, for a given $w \in C^\infty(\partial_+ SM, \mathbb{C})$ we denote by $w^\sharp : SM \rightarrow \mathbb{C}$ the unique solution to the transport equation

$$Xw^\sharp + aw^\sharp = 0 \quad (SM), \quad w^\sharp|_{\partial_+ SM} = w.$$

We then define $Q_a : C(\partial_+ SM, \mathbb{C}) \rightarrow C(\partial SM, \mathbb{C})$, by $Q_a w := w^\sharp|_{\partial SM}$. Q_a takes the expression

$$Q_a w(x, v) = \begin{cases} w(x, v) & (x, v) \in \partial_+ SM, \\ \exp\left(-\int_{-\tau(x, -v)}^0 a(\gamma_{x, v}(t))\right) w(\alpha(x, v)) & (x, v) \in \partial_- SM, \end{cases} \quad (2)$$

where α denotes the *scattering relation*¹ defined in Section 3. As $Q_a w$ may only be continuous even when $w \in C^\infty(\partial_+ SM)$, we define

$$\mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C}) := \{w \in C^\infty(\partial_+ SM, \mathbb{C}), Q_a w \in C^\infty(\partial SM)\}.$$

We also introduce the operator $B_a : C(\partial SM, \mathbb{C}) \rightarrow C(\partial_+ SM, \mathbb{C})$ by

$$B_a u(x, v) := \exp\left(\int_0^{\tau(x, v)} a(\gamma_{x, v}(t)) dt\right) u \circ \alpha(x, v) - u(x, v), \quad (x, v) \in \partial_+ SM. \quad (3)$$

Next, we introduce the operator $P_a : \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C}) \rightarrow C^\infty(\partial_+ SM, \mathbb{C})$ defined by $P_a := B_a H Q_a$, where H is the fiberwise Hilbert transform, defined in Section 3. Clearly the operator P_a is completely determined by the scattering relation α and the unattenuated ray transform of a . The first main result of the paper is that the operator P_a characterizes the ray transform \mathcal{I}_a over pairs.

Theorem 2.1. *Let (M, g) be a simple surface and let $a \in C^\infty(M, \mathbb{R})$. Then a function $u \in C^\infty(\partial_+ SM, \mathbb{C})$ belongs to the range of \mathcal{I}_a if and only if $u = P_a w$ for some $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$.*

As mentioned in the introduction, the corresponding operator $P := P_0$, first introduced in [26], splits into two operators P_+ and P_- which characterize the ranges of I^0 and I^1 separately. In the attenuated case, such a splitting is no longer obvious. Sitting within the range of \mathcal{I}_a , a first range characterization for I_a^0 and I_a^1

¹Throughout the paper, α may denote either the scattering relation, or a general one-form, though which occurrence it is should appear clear from the context.

can be obtained by adding conditions on the preimage by P_a . Before stating the result, we introduce some notations.

Since M is oriented there is a circle action on the fibres of SM with infinitesimal generator V called the vertical vector field. For any two functions $u, v : SM \rightarrow \mathbb{C}$ define an inner product:

$$\langle u, v \rangle_{L^2(SM)} = \int_{SM} u \bar{v} d\Sigma^3,$$

where $d\Sigma^3$ is the Liouville measure of g on SM . The space $L^2(SM, \mathbb{C})$ decomposes orthogonally as a direct sum

$$L^2(SM, \mathbb{C}) = \bigoplus_{k \in \mathbb{Z}} H_k$$

where H_k is the eigenspace of $-iV$ corresponding to the eigenvalue k . Any function $u \in C^\infty(SM, \mathbb{C})$ has a Fourier series expansion

$$u = \sum_{k=-\infty}^{\infty} u_k, \quad u_k \in \Omega_k := C^\infty(SM, \mathbb{C}) \cap H_k.$$

In particular, $u \mapsto u_0$ and $u \mapsto u_{-1} + u_1$ are the projections of functions on SM onto functions and 1-forms on M , respectively; see Section 3.2.

Theorem 2.2. *Let (M, g) be a simple surface and let $a \in C^\infty(M, \mathbb{C})$. The following range characterizations hold:*

- (1) *A function $u \in C^\infty(\partial_+ SM, \mathbb{C})$ belongs to the range of I_a^0 if and only if $u = P_a w$ for some $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ such that $w_0^\# = 0$.*
- (2) *A function $u \in C^\infty(\partial_+ SM, \mathbb{C})$ belongs to the range of I_a^1 acting on solenoidal one-forms if and only if $u = P_a w$ for some $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ such that $w_{-1}^\# + w_1^\# = dp$ for some $p \in C^\infty(M, \mathbb{C})$.*

We now derive reconstruction formulas for pairs, which in turn yield a second range characterization. Since the transform over pairs \mathcal{I}_a has a kernel (namely, the “ a -potential” pairs), it is first useful to change its domain in such a way which makes it injective without altering its range. To this end, we make our way in Theorem 6.1 and Lemma 6.2 and 6.3, into proving that any element $\mathcal{D} \in \text{Range } \mathcal{I}_a$ decomposes uniquely as

$$\mathcal{D} = \mathcal{I}_a[\star dh_0 + \omega_1 + \omega_{-1}, f] = I_a^0 f + I_a^\perp h_0 + I_a^{+1} \omega_1 + I_a^{-1} \omega_{-1},$$

with $f \in C^\infty(M)$, $h_0 \in C_0^\infty(M)$ and $\omega_{\pm 1} \in \ker^{\pm 1} \eta_\mp$ are holomorphic and anti-holomorphic one-forms. Moreover, $\mathcal{D} = 0$ if and only if f, h_0, ω_1 and ω_{-1} vanish identically. This suggests that the quadruple $(f, h_0, \omega_1, \omega_{-1})$ can be reconstructed from \mathcal{D} , and we proceed to provide reconstruction formulas for each term in Section 7. We first reconstruct ω_1 and ω_{-1} from \mathcal{D} in Theorem 7.3, and in turn, explain how to remove $\mathcal{I}_a[\omega_1 + \omega_{-1}, 0]$ from \mathcal{D} . This is done using Hilbert bases of square integrable harmonic one-forms, combined with integration by parts on SM using appropriate adjoint transport solutions with one-sided fiber-harmonic content. In fact, the reconstruction of the terms $\omega_{\pm 1}$ is new even in the unattenuated case,

for which the first inversion formulas for one-forms appearing in [26] only treated one-forms $\omega = \star dh$ with $h|_{\partial M} = 0$.

After reconstructing ω_1 and ω_{-1} , it remains to reconstruct (f, h_0) from $\mathcal{I}_a[\star dh_0, f]$. As a means to obtain exact reconstruction formulas (i.e., not up to Fredholm errors), we first construct in section 7.2 a “holomorphization operator” $\vec{\mathcal{B}} : C^\infty(\partial SM) \rightarrow C^\infty(\partial_+ SM)$ (see Theorem 7.5 for details) such that if the equation $Xu = -f$ holds with f holomorphic, then the function $\vec{u} = u - (\vec{\mathcal{B}}(u|_{\partial SM})_\psi)$ is a holomorphic solution of $X\vec{u} = -f$ with \vec{u}_0 constant. An antiholomorphization counterpart $\overleftarrow{\mathcal{B}}$ is also defined there. Such operators, which allow to extract holomorphic and antiholomorphic contents at will, together with the use of so-called holomorphic and anti-holomorphic integrating factors first defined in [32], are key to deriving the following reconstruction formulas, which we prove in Section 7.3. See Section 3.2 for a definition of the Guillemin-Kazhdan operators η_\pm appearing below.

Theorem 2.3. *Let (M, g) a simple surface and $a \in C^\infty(M, \mathbb{C})$. Define \vec{w} and \overleftarrow{w} smooth holomorphic and antiholomorphic, odd, solutions of $X\vec{w} = X\overleftarrow{w} = -a$, and let $\vec{\mathcal{B}}$ and $\overleftarrow{\mathcal{B}}$ as in Theorem 7.5 and Corollary 7.6. Then the functions $(h_0, f) \in C_0^\infty(M) \times C^\infty(M)$ can be reconstructed from data $\mathcal{I} := \mathcal{I}_a[\star dh_0, f]$ (extended by zero on $\partial_- SM$) via the following formulas:*

$$f = -\eta_+ \vec{\mathcal{D}}_{-1} - \eta_- \overleftarrow{\mathcal{D}}_1 - \frac{a}{2} (\vec{\mathcal{D}}_0 + \overleftarrow{\mathcal{D}}_0 + i(g_+ - g_-)),$$

$$h_0 = \frac{1}{2}(g_+ + g_-) - \frac{i}{2}(\vec{\mathcal{D}}_0 - \overleftarrow{\mathcal{D}}_0),$$

where we have defined $\vec{\mathcal{D}} := e^{\vec{w}}(\vec{\mathcal{B}}(\mathcal{I}e^{-\vec{w}}|_{\partial SM}))_\psi$, $\overleftarrow{\mathcal{D}} := e^{\overleftarrow{w}}(\overleftarrow{\mathcal{B}}(\mathcal{I}e^{-\overleftarrow{w}}|_{\partial SM}))_\psi$, and where $g_\pm \in \ker^0 \eta_\pm$, uniquely characterized by their boundary conditions

$$g_+|_{\partial M} = -i(\mathcal{I} - \vec{\mathcal{D}}|_{\partial SM})_0, \quad g_-|_{\partial M} = i(\mathcal{I} - \overleftarrow{\mathcal{D}}|_{\partial SM})_0.$$

The reconstruction formulas above then allow to construct in (31) and (38) explicit linear, idempotent operators $P_{a,0}, P_{a,\perp}, P_{a,\pm 1} : \text{Range } \mathcal{I}_a \rightarrow \text{Range } \mathcal{I}_a$, such that

$$P_{a,\pm 1}\mathcal{D} = I_a^{\pm 1}\omega_{\pm 1}, \quad P_{a,0}\mathcal{D} = I_a^0 f, \quad P_{a,\perp}\mathcal{D} = I_a^\perp h_0.$$

Such operators allow to establish the following range characterization:

Theorem 2.4. *Let (M, g) a simple surface and let $a \in C^\infty(M, \mathbb{C})$. Then the following hold:*

- (i) *A function $u \in C^\infty(\partial_+ SM, \mathbb{C})$ belongs to the range of I_a^0 if and only if $u = P_a w$ for some $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ and $P_{a,1}u = P_{a,-1}u = P_{a,\perp}u = 0$.*
- (ii) *A function $u \in C^\infty(\partial_+ SM, \mathbb{C})$ belongs to the range of I_a^1 acting on solenoidal one-forms if and only if $u = P_a w$ for some $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ and $P_{a,0}u = 0$.*

This characterization is of practical relevance as it allows to project noisy data onto the range of I_a^0 or I_a^1 acting on solenoidal one-forms using explicit operators, before inversion.

Outline and roadmap of proofs. We first study the space of pairs [one-form, function] in Section 3.3, on which the operator \mathcal{I}_a is defined. Proving Theorem 2.1 is based on the factorization $-2\pi P_a = \mathcal{I}_a \begin{bmatrix} 0 & \star^d \\ \star_d & 0 \end{bmatrix} \mathcal{I}_{-\bar{a}}^*$, which completes the proof once the surjectivity of $\mathcal{I}_{-\bar{a}}^*$ is proved in appropriate functional settings. Such a surjectivity mainly relies on the injectivity of $\mathcal{I}_{-\bar{a}}$ ([32, Theorem 1.2]), and is based on pseudodifferential arguments on a slightly extended surface. Theorem 2.2 then follows by finding the appropriate additional conditions which characterize each sub-range.

On to the proof of Theorem 2.4, we first explain in Section 6 how to change the domain \mathcal{I}_a in a way which makes it injective, in particular via the mapping $(f, h_0, \omega_1, \omega_{-1}) \mapsto \mathcal{I}_a[\star dh_0 + \omega_1 + \omega_{-1}, f]$. Section 7 then explains how to reconstruct each term: we first reconstruct ω_1 and ω_{-1} in Section 7.1; then introduce holomorphization operators in Section 7.2; finally, we provide reconstruction formulas for (f, h_0) in Section 7.3. In both sections 7.1 and 7.3, we explain the implications of such inversions on the ability to construct projection operators for Theorem 2.4.

3. PRELIMINARIES

3.1. Scattering relation and transport equations. Recall that for $(x, v) \in SM$, $\tau(x, v)$ denotes the first non-negative exit time $\tau(x, v)$ of the geodesic $\gamma_{x,v}$, with $x = \gamma_{x,v}(0)$, $v = \dot{\gamma}_{x,v}(0)$. The *scattering relation* is the map $\alpha : \partial SM \rightarrow \partial SM$ defined as

$$\alpha(x, v) = \varphi_{\pm\tau(x, \pm v)}(x, v), \quad (x, v) \in \partial_{\pm} SM,$$

Since (M, g) is assumed to be simple, by [35, Lemma 4.1.1] we conclude that the scattering relation α is diffeomorphism and $\alpha^2 = \text{Id}$.

The attenuated ray transform (1) can be realized as the trace on $\partial_+ SM$ of the solution $u : SM \rightarrow \mathbb{C}$ to the following transport equation on SM ,

$$Xu + au = -f \quad (SM), \quad u|_{\partial_- SM} = 0,$$

where $f \in C^\infty(SM)$ represents the “source term”. This equation has a unique solution u^f , since on any fixed geodesic the transport equation is an ODE with zero initial condition and an integral expression gives us that $u|_{\partial_+ SM}$ matches (1). For $w \in C^\infty(\partial_+ SM, \mathbb{C}^n)$ given, let us denote $w_\psi(x, v) := w(\varphi_{-\tau(x, -v)}(x, v))$ the unique solution u to the transport problem

$$Xu = 0 \quad (SM), \quad u|_{\partial_+ SM} = w.$$

For $a \in C^\infty(M, \mathbb{C})$, define the integrating factor $U_a : SM \rightarrow \mathbb{C}$, unique solution to

$$(X + a)U_a = 0 \quad (SM), \quad U_a|_{\partial_+ SM} = 1,$$

whose integral expression is given by

$$U_a(x, v) = \exp \left(- \int_{-\tau(x, -v)}^0 a(\gamma_{x,v}(s)) ds \right), \quad (x, v) \in SM.$$

By solving explicitly the transport equation along the geodesic, one can show that

$$U_a(\varphi_t(x, v)) = \exp \left(- \int_0^t a(\gamma_{x,v}(s)) ds \right), \quad (x, v) \in SM,$$

and hence the following integral formula holds:

$$I_a f(x, v) = \int_0^{\tau(x, v)} U_a^{-1}(\varphi_t(x, v)) f(\varphi_t(x, v)) dt, \quad (x, v) \in \partial_+ SM.$$

With the U_a notation, notice that the function w^\sharp defined in Section 2 is nothing but $w^\sharp(x, v) = U_a(x, v)w_\psi(x, v)$, and Q_a defined in (2) takes the expression

$$Q_a w(x, v) := \begin{cases} w(x, v) & (x, v) \in \partial_+ SM, \\ U_a(x, v)(w \circ \alpha)(x, v) & (x, v) \in \partial_- SM. \end{cases}$$

The space of those w for which w^\sharp is smooth in SM is denoted by

$$\begin{aligned} \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C}) &:= \{w \in C^\infty(\partial_+ SM, \mathbb{C}) : w^\sharp \in C^\infty(SM, \mathbb{C})\} \\ &= \{w \in C^\infty(\partial_+ SM, \mathbb{C}) : Q_a w \in C^\infty(\partial(SM), \mathbb{C})\}, \end{aligned}$$

where the second equality is a characterization in terms of the operator Q_a , proved in [24, Lemma 5.1].

Another characterization of the B_a operator defined in (3) is that, for any smooth function $\psi : SM \rightarrow \mathbb{C}$, we have

$$\begin{aligned} I_a((X + a)\psi)(x, v) &= \exp\left(\int_0^{\tau(x, v)} a(\gamma_{x, v}(t)) dt\right) \psi \circ \alpha(x, v) - \psi(x, v) \\ &= B_a \psi|_{\partial SM}(x, v). \end{aligned}$$

3.2. Geometry and Fourier analysis on SM . Since M is oriented there is a circle action on the fibres of SM with infinitesimal generator V called the vertical vector field. We complete X, V to a global frame of $T(SM)$ by defining the vector field $X_\perp := [X, V]$, where $[\cdot, \cdot]$ is the Lie bracket for vector fields. For any two functions $u, v : SM \rightarrow \mathbb{C}$ define an inner product:

$$\langle u, v \rangle_{L^2(SM)} = \int_{SM} u \bar{v} d\Sigma^3,$$

where $d\Sigma^3$ is the Liouville measure of g on SM . The space $L^2(SM, \mathbb{C})$ decomposes orthogonally as a direct sum

$$L^2(SM, \mathbb{C}) = \bigoplus_{k \in \mathbb{Z}} H_k$$

where H_k is the eigenspace of $-iV$ corresponding to the eigenvalue k . Any function $u \in C^\infty(SM, \mathbb{C})$ has a Fourier series expansion

$$u = \sum_{k=-\infty}^{\infty} u_k, \quad u_k \in \Omega_k := C^\infty(SM, \mathbb{C}) \cap H_k.$$

We recall the first order elliptic operators due to Guillemin and Kazhdan [10], defined by $\eta_\pm = \frac{1}{2}(X \pm iX_\perp)$. By the commutation relations $[-iV, \eta_+] = \eta_+$ and $[-iV, \eta_-] = -\eta_-$ we see that

$$\eta_+ : \Omega_k \rightarrow \Omega_{k+1}, \quad \eta_- : \Omega_k \rightarrow \Omega_{k-1}.$$

For the sequel, let us denote, for any $k \in \mathbb{Z}$, $\ker^k \eta_\pm := \Omega_k \cap \ker \eta_\pm$.

An important tool in our approach is the *fiberwise Hilbert transform* $H : C^\infty(SM, \mathbb{C}) \rightarrow C^\infty(SM, \mathbb{C})$, which we define in terms of Fourier coefficients as

$$H(u_k) = -i \operatorname{sgn}(k) u_k, \quad (\text{with the convention } \operatorname{sgn}(0) = 0).$$

The following commutator formula, which was derived by Pestov and Uhlmann in [27] and generalized in [24], will play an important role.

$$[H, X + a]u = X_\perp u_0 + (X_\perp u)_0, \quad u \in C^\infty(SM, \mathbb{C}). \quad (4)$$

This formula has been frequently used in recent works on inverse problems, see [22, 23, 24, 26, 27, 32].

3.3. The space of pairs. The inner product in the space $\mathcal{L}^2(M, \mathbb{C})$ is given by

$$([\alpha, f] \mid [\beta, h])_{\mathcal{L}^2(M, \mathbb{C})} = \int_M \langle \alpha, \bar{\beta} \rangle_g d\operatorname{Vol}_g + \int_M f \bar{h} d\operatorname{Vol}_g. \quad (5)$$

Assume that $a \in C^\infty(M, \mathbb{C})$. Consider the following operators $d_a : H^1(M, \mathbb{C}) \rightarrow \mathcal{L}^2(M, \mathbb{C})$ and $\delta_a : \mathcal{H}^1(M, \mathbb{C}) \rightarrow L^2(M, \mathbb{C})$ defined by

$$d_a h = [dh, ah], \quad \delta_a [\alpha, f] = \delta \alpha - \bar{a} f.$$

The following integration by parts formula holds for these operators:

$$(\delta_a [\alpha, f] \mid h)_{L^2(M, \mathbb{C})} + ([\alpha, f] \mid d_a h)_{\mathcal{L}^2(M, \mathbb{C})} = (i_\nu \alpha \mid h)_{L^2(\partial M)},$$

where ν is the outward unit normal on ∂M . In particular, we obtain $d_a^* = -\delta_a$. Introducing the spaces of *a-solenoidal* and *a-potential* pairs

$$\begin{aligned} \mathcal{L}_{a, \text{sol}}^2(M, \mathbb{C}) &= \{[\alpha, f] \in \mathcal{L}^2(M, \mathbb{C}) : \delta_a [\alpha, f] = 0\}, \\ \mathcal{L}_{a, \text{pot}}^2(M, \mathbb{C}) &= \{d_a h : h \in H_0^1(M, \mathbb{C})\}, \end{aligned}$$

Proposition 3.1 below implies the \mathcal{L}^2 -orthogonal decompositions

$$\begin{aligned} \mathcal{L}^2(M, \mathbb{C}) &= \mathcal{L}_{a, \text{sol}}^2(M, \mathbb{C}) \oplus \mathcal{L}_{a, \text{pot}}^2(M, \mathbb{C}), \\ \mathcal{C}^\infty(M, \mathbb{C}) &= \mathcal{C}_{a, \text{sol}}^\infty(M, \mathbb{C}) \oplus \mathcal{C}_{a, \text{pot}}^\infty(M, \mathbb{C}), \end{aligned} \quad (6)$$

where we have defined $\mathcal{C}_{a, \text{sol/pot}}^\infty(M, \mathbb{C}) := \mathcal{L}_{a, \text{sol/pot}}^2(M, \mathbb{C}) \cap \mathcal{C}^\infty(M, \mathbb{C})$.

Proposition 3.1. *Let $a \in C^\infty(M, \mathbb{C})$ and let $k \geq 0$ be an integer. For a given $[\alpha, f] \in \mathcal{H}^k(M, \mathbb{C})$ there are unique $[\beta, h] \in \mathcal{H}^k(M, \mathbb{C})$ and $b \in H^{k+1}(M, \mathbb{C}) \cap H_0^1(M, \mathbb{C})$ such that $[\alpha, f] = [\beta, h] + d_a b$ and $\delta_a [\beta, h] = 0$. Moreover, if $[\alpha, f] \in \mathcal{C}^\infty(M, \mathbb{C})$ then $[\beta, h] \in \mathcal{C}_{a, \text{sol}}^\infty(M, \mathbb{C})$ and $b \in C^\infty(M, \mathbb{C})$ with $b|_{\partial M} = 0$.*

Proof. For $[\alpha, f] \in \mathcal{L}^2$, consider the problem for $b \in H_0^1(M, \mathbb{C})$

$$-\delta_a d_a b = -\delta_a [\alpha, f] \in H^{-1}(M, \mathbb{C}), \quad b|_{\partial M} = 0,$$

whose weak formulation consists in finding $b \in H_0^1(M, \mathbb{C})$ such that,

$$(d_a b, d_a b')_{\mathcal{L}^2(M, \mathbb{C})} = \langle -\delta_a [\alpha, f], b' \rangle_{H^{-1}, H_0^1}, \quad \forall b' \in H_0^1(M, \mathbb{C}),$$

where the sesquilinear form on the left-hand side, given by

$$(d_a b, d_a b')_{\mathcal{L}^2(M, \mathbb{C})} = \int_M \langle d_a b, \overline{d_a b'} \rangle_g d\operatorname{Vol}_g + \int_M |a|^2 b \bar{b'} d\operatorname{Vol}_g,$$

is hermitian, continuous and coercive (since, when $b = b'$, the second term is nonnegative and the first term controls the H_0^1 norm by virtue of Poincaré's inequality). The existence and uniqueness of such a b is then provided by Lax-Milgram's theorem, see e.g. [7, Theorem 1, Sec. 6.2.1]. Once b is constructed, set $[\beta, h] = [\alpha, f] - d_a b$ and the \mathcal{L}^2 decomposition follows. Moreover, following results on higher order regularity for solutions of strongly elliptic equations (see for example [39, Proposition 11.10]), if $[\alpha, f] \in \mathcal{H}^k$, then $b \in H^{k+1} \cap H_0^1$ and thus $[\beta, h] \in \mathcal{H}^k$. In particular, if $[\alpha, f]$ are smooth, so are b and $[\beta, h]$. \square

3.4. Extension operators for a -solenoidal pairs. Our aim in this subsection is to extend a -solenoidal pair to a larger manifold as compactly supported a -solenoidal pair in the \mathcal{C}^∞ setting. We will follow the arguments of [13] and [25].

Here and in what follows, $\mathcal{H}_{U,a,\text{sol}}^1(\widetilde{M}^{\text{int}}, \mathbb{C})$ and $\mathcal{C}_{U,a,\text{sol}}^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})$ denote the subspaces of $\mathcal{H}_{a,\text{sol}}^1(\widetilde{M}^{\text{int}}, \mathbb{C})$ and $\mathcal{C}_{a,\text{sol}}^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})$, respectively, consisting of elements supported in U .

We start with the following lemma on the existence of a -solenoidal extensions that might not be compactly supported.

Lemma 3.2 (Smooth a -solenoidal extensions). *Let M be a compact simply connected manifold contained in the interior of some Riemannian manifold (\widetilde{M}, g) and let $a \in C^\infty(\widetilde{M}, \mathbb{C})$. There is an open neighborhood U of M and a linear operator $\mathcal{E}_{a,U} : \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C}) \rightarrow \mathcal{C}_{a,\text{sol}}^\infty(U, \mathbb{C})$ with $\mathcal{E}_{a,U} = \text{Id}$ on M and $\|\mathcal{E}_{a,U}[\alpha, f]\|_{\mathcal{H}^1(U, \mathbb{C})} \leq C\|[\alpha, f]\|_{\mathcal{H}^1(M, \mathbb{C})}$.*

Proof. We cover ∂M in \widetilde{M} by charts $\{(\mathcal{O}_\kappa, \Theta_\kappa)\}_\kappa$ with semi-geodesic local coordinates, i.e. each coordinate map $\Theta_\kappa : \mathcal{O}_\kappa \rightarrow \mathbb{R}^n$ is of the form $\Theta_\kappa(p) = (x^1, \dots, x^{n-1}, x^n) = (x', x^n)$ such that $\Theta_\kappa^{-1}(\{x^n = 0\}) \cap \mathcal{O}_\kappa \subset \partial M$, $\Theta_\kappa^{-1}(\{x^n < 0\}) \cap \mathcal{O}_\kappa \subset M^{\text{int}}$ and $(\Theta_\kappa^{-1})_* \partial_n = \nu$ is the unit outward (from M) normal to ∂M . In these coordinates, we have

$$g^{kn} = \delta^{kn}, \quad \Gamma_{kn}^n = \Gamma_{nn}^k = 0, \quad k = 1, \dots, n.$$

We determine $U \setminus M$ as the sufficiently small semi-geodesic neighborhood of ∂M in \widetilde{M} such that $\overline{U} \subset \cup_\kappa \mathcal{O}_\kappa$.

Given $[\alpha, f] \in \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$. We extend the function f and the components $\alpha_{i'}$, $i' = 1, \dots, n-1$, smoothly to U , and denote the extensions by h and $\beta_{i'}$, respectively. By the results in [33], these extensions can be done in a stable way

$$\|h\|_{H^1(U, \mathbb{C})} \leq C\|f\|_{H^1(M, \mathbb{C})}, \quad \|\beta_{i'}\|_{H^1(U, \mathbb{C})} \leq C\|\alpha_{i'}\|_{H^1(M, \mathbb{C})}, \quad i' = 1, \dots, n-1. \quad (7)$$

Now we construct the last component β_n in $\Theta_\kappa^{-1}(\{x^n > 0\}) \cap \mathcal{O}_\kappa$. Since we want a -solenoidal extension, writing $h^\kappa = h \circ \Theta_\kappa^{-1}$ and $\beta_i^\kappa = \beta_i \circ \Theta_\kappa^{-1}$, $i = 1, \dots, n$, we have

$$\partial_n \beta_n^\kappa - \sum_{j,k < n} g^{jk} \Gamma_{jk}^n \beta_n^\kappa = \bar{a} h^\kappa - \sum_{j,k < n} g^{jk} \partial_j \beta_k^\kappa + \sum_{j,k,l < n} g^{jk} \Gamma_{jk}^l \beta_l^\kappa. \quad (8)$$

Observe that the right side is known, so it is a first order linear ordinary differential equation. Given the initial condition $\beta_n^\kappa(x', 0) = \alpha_n \circ \Theta_\kappa^{-1}(x', 0)$, there is a unique

solution $\beta_n^\kappa(x', x^n)$. In this way we construct continuous β_n^κ in $\{x^n > 0\} \cap \Theta_\kappa(\mathcal{O}_\kappa)$ which depends smoothly on x' . If U is sufficiently close to M , one can show that in each chart \mathcal{O}_κ the following holds

$$|\beta_n^\kappa(x', x^n)|^2 \leq C_{\mathcal{O}_\kappa} \left(|\alpha_n \circ \Theta_\kappa^{-1}(x', 0)|^2 + |h^\kappa(x', x^n)|^2 + \sum_{j,k < n} |\partial_j \beta_k^\kappa(x', x^n)|^2 \right),$$

for all $(x', x^n) \in \{x^n > 0\} \cap \Theta_\kappa(\mathcal{O}_\kappa \cap U)$. Integrating over U and using compactness of M , we obtain

$$\|\beta_n\|_{L^2(U, \mathbb{C})} \leq C \left(\|\alpha_n\|_{L^2(\partial M, \mathbb{C})} + \|f\|_{H^1(M, \mathbb{C})} + \sum_{j < n} \|\alpha_j\|_{H^1(M, \mathbb{C})} \right).$$

Let V be a neighborhood of ∂M in M . Then for all $(x', x^n) \in \{x^n < 0\} \cap \Theta_\kappa(\mathcal{O}_\kappa \cap V)$ we can show that

$$|\alpha_n(x', 0)|^2 \leq C_{\mathcal{O}_\kappa} \left(|\alpha_n(x', x^n)|^2 + \int_{x^n}^0 |\alpha_n(x', x^n)|^2 dx^n + \int_{x^n}^0 |\partial_n \alpha_n(x', x^n)|^2 dx^n \right).$$

Integrating over M and using compactness of M , we obtain

$$\|\alpha_n\|_{L^2(\partial M, \mathbb{C})} \leq C \|\alpha\|_{H^1(M, \mathbb{C})},$$

and hence

$$\|\beta_n\|_{L^2(U, \mathbb{C})} \leq C \left(\|f\|_{H^1(M, \mathbb{C})} + \|\alpha\|_{H^1(M, \mathbb{C})} \right).$$

Therefore, combining the latter inequality together with (7) and (8), we come to

$$\|[\beta, h]\|_{\mathcal{H}^1(U, \mathbb{C})} \leq C \|[\alpha, f]\|_{\mathcal{H}^1(M, \mathbb{C})}.$$

Now, we want to show the smoothness of β_n . Differentiating (8) with respect to x^n , we show that $\beta_n^\kappa(x', x^n)$ is smooth in $\{x^n \geq 0\} \cap \Theta_\kappa(\mathcal{O}_\kappa)$. Moreover, using (8) and induction on the order of derivative with respect to x^n , we show

$$\partial_n^m \beta_n^\kappa(x', 0) = \partial_n^m \alpha_n \circ \Theta_\kappa^{-1}(x', 0)$$

for all $m \geq 0$ integers. In other words, $\partial_n^m \beta_n(x', 0)$ agree with $\partial_n^m \alpha_n \circ \Theta_\kappa^{-1}(x', 0)$. Therefore, we obtain a smooth a -solenoidal pair

$$\mathcal{E}_{a,U}[\alpha, f] = \begin{cases} [\alpha, f] & \text{on } M, \\ [\beta, h] & \text{on } U \setminus M, \end{cases}$$

with $\|\mathcal{E}_{a,U}[\alpha, f]\|_{\mathcal{H}^1(U, \mathbb{C})} \leq C \|[\alpha, f]\|_{\mathcal{H}^1(M, \mathbb{C})}$. \square

Proposition 3.3. *Let M be a compact simply connected manifold contained in the interior of some Riemannian manifold (\widetilde{M}, g) and let $a \in C^\infty(\widetilde{M}, \mathbb{C})$. There is a precompact neighborhood W of M in $\widetilde{M}^{\text{int}}$ and a bounded map $\mathcal{E}_a : \mathcal{H}_{a,\text{sol}}^1(M, \mathbb{C}) \rightarrow \mathcal{H}_{W,a,\text{sol}}^1(\widetilde{M}^{\text{int}}, \mathbb{C})$ such that $\mathcal{E}_a|_M = \text{Id}$. Moreover, the restriction of \mathcal{E}_a to $\mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$ maps $\mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$ into $\mathcal{C}_{W,a,\text{sol}}^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})$.*

Proof. Given $[\alpha, f] \in \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$, Lemma 3.2 implies the existence of a neighborhood U of M and a linear operator $\mathcal{E}_{a,U} : \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C}) \rightarrow \mathcal{C}_{a,\text{sol}}^\infty(U, \mathbb{C})$ with $\mathcal{E}_{a,U} = \text{Id}$ on M and

$$\|\mathcal{E}_{a,U}[\alpha, f]\|_{\mathcal{H}^1(U, \mathbb{C})} \leq C \|[\alpha, f]\|_{\mathcal{H}^1(M, \mathbb{C})}.$$

Consider an open precompact set W such that $U \subset \overline{U} \subset W \subset \overline{W} \subset \widetilde{M}^{\text{int}}$. Then, using [33], we extend $\mathcal{E}_{a,U}[\alpha, f]$ to $\widetilde{M}^{\text{int}}$ and multiply the extension by a smooth cut-off function which is equal to 1 in U and supported in W , and we denote the resultant pair by $[\beta, h]$. Combining the result of [33] and Lemma 3.2 implies that

$$\|[\beta, h]\|_{\mathcal{H}^1(\widetilde{M}^{\text{int}}, \mathbb{C})} \leq C\|[\alpha, f]\|_{\mathcal{H}^1(M, \mathbb{C})}.$$

Set $w = \delta_a[\beta, h]$ and $D = W \setminus \overline{M}$. We have $\text{supp } w \subset W \setminus U \subset D$.

We claim that $(w|v)_{L^2(D, \mathbb{C})} = 0$ for all $v \in H^1(D, \mathbb{C}) \cap \ker d_a$. Then, by [6, Corollary 1.6] (see also [28, Section 5.1]), there is a smooth one-form β_D on $\widetilde{M}^{\text{int}}$ such that $\delta\beta_D = -w$ and $\text{supp } \beta_D \subset W \setminus M^{\text{int}}$. Moreover, β_D satisfies

$$\|\beta_D\|_{H^1(\widetilde{M}^{\text{int}}, \mathbb{C})} \leq C\|w\|_{L^2(\widetilde{M}^{\text{int}}, \mathbb{C})}.$$

We define

$$\mathcal{E}_a[\alpha, f] = [\beta + \beta_D, h].$$

Then $\mathcal{E}_a[\alpha, f]|_M = [\alpha, f]$, $\text{supp } \mathcal{E}_a[\alpha, f] \subset W$ and $\delta_a \mathcal{E}_a[\alpha, f] = \delta_a[\beta, h] + \delta\beta_D = w - w = 0$. Hence $\mathcal{E}_a[\alpha, f] \in \mathcal{C}_{W,a,\text{sol}}^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})$ and

$$\|\mathcal{E}_a[\alpha, f]\|_{\mathcal{H}^1(\widetilde{M}^{\text{int}}, \mathbb{C})} \leq C\|[\alpha, f]\|_{\mathcal{H}^1(M, \mathbb{C})}.$$

Since $\mathcal{C}^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})$ is dense in $\mathcal{H}^1(\widetilde{M}^{\text{int}}, \mathbb{C})$ under the \mathcal{H}^1 -norm, we extend \mathcal{E}_a to a bounded map $\mathcal{E}_a : \mathcal{H}_{a,\text{sol}}^1(M, \mathbb{C}) \rightarrow \mathcal{H}_{W,a,\text{sol}}^1(\widetilde{M}^{\text{int}}, \mathbb{C})$ with $\mathcal{E}_a|_M = \text{Id}$.

Now it is left to prove that $(w|v)_{L^2(D, \mathbb{C})} = 0$ for all $v \in H^1(D, \mathbb{C}) \cap \ker d_a$. For this, we study the solutions of the homogeneous problem. Let $v \in H^1(D, \mathbb{C})$ be a solution of

$$\begin{cases} (-\Delta_g + |a|^2)v = 0 \text{ in } D, \\ \partial_N v = 0 \text{ on } \partial D, \end{cases}$$

where N is the unit outward normal on ∂D . Then by Green's formula

$$\|\nabla v\|_{L^2(D, \mathbb{C})}^2 + \|av\|_{L^2(D, \mathbb{C})}^2 = ((-\Delta_g + |a|^2)v|v)_{L^2(D, \mathbb{C})} + (\partial_N v|v)_{L^2(\partial D, \mathbb{C})} = 0.$$

Thus, $\nabla v \equiv 0$ and $av \equiv 0$. In other words, $v \in \ker d_a$. Let K_a denotes the set of the solutions of the homogeneous problem, then

$$K_a = \{v \in H^1(D, \mathbb{C}) : \nabla v \equiv 0, av \equiv 0\} = H^1(D, \mathbb{C}) \cap \ker d_a.$$

If $a \equiv 0$, then K_a consists of constant solutions. Hence, for $v = \text{const} \in K_a$, integration by parts gives

$$\begin{aligned} (w|v)_{L^2(D, \mathbb{C})} &= (\delta_a[\beta, h]|v)_{L^2(D, \mathbb{C})} = -(i_\nu \alpha|v)_{L^2(\partial M, \mathbb{C})} \\ &= -(\delta_a[\alpha, f]|v)_{L^2(M, \mathbb{C})} - ([\alpha, f]|d_a v)_{L^2(M, \mathbb{C})} = 0. \end{aligned}$$

Now, if $a \not\equiv 0$, then $K_a = \{0\}$. Hence, for $v = 0 \in K_a$ we trivially have $(w|v)_{L^2(D, \mathbb{C})} = (w|0)_{L^2(D, \mathbb{C})} = 0$. \square

4. SURJECTIVITY RESULTS FOR THE ADJOINTS

The main purpose of this section is to obtain the surjectivity theorem 4.2, of the adjoint \mathcal{I}_a^* , on which our range characterizations hinge. We first compute the adjoints of I_a and \mathcal{I}_a in section 4.1, then prove Theorem 4.2 in section 4.2.

4.1. Adjoints of I_a and \mathcal{I}_a . Let $d\Sigma^2$ be the volume form on $\partial(SM)$. Denote by $L_\mu^2(\partial_+ SM, \mathbb{C})$ the completion of $C_c^\infty(\partial_+ SM, \mathbb{C})$ for the inner product

$$\langle h, h' \rangle_{L_\mu^2(\partial_+ SM, \mathbb{C})} = \int_{\partial_+ SM} h \overline{h'} \mu \, d\Sigma^2, \quad \mu(x, v) := \langle v, \nu_x \rangle_{g(x)}.$$

As in [23], using the integral representation for I_a and Santaló formula [4, Lemma A.8], one can show that I_a can be extended to a bounded operator $I_a : L^2(SM, \mathbb{C}) \rightarrow L_\mu^2(\partial_+ SM, \mathbb{C})$.

Consider the dual $I_a^* : L_\mu^2(\partial_+ SM, \mathbb{C}) \rightarrow L^2(SM, \mathbb{C})$ of I_a , for which we now find an expression. For this consider $h \in L_\mu^2(\partial_+ SM, \mathbb{C})$. Using Santaló's formula, for $f \in L^2(SM, \mathbb{C})$, we compute

$$\begin{aligned} \langle I_a f, h \rangle_{L_\mu^2(\partial_+ SM, \mathbb{C})} &= \int_{\partial_+ SM} \left(\int_0^{\tau(x, v)} U_a^{-1}(\varphi_t(x, v)) f(\varphi_t(x, v)) dt \right) \overline{h(x, v)} \mu \, d\Sigma^2 \\ &= \int_{\partial_+ SM} \left(\int_0^{\tau(x, v)} U_a^{-1}(\varphi_t(x, v)) f(\varphi_t(x, v)) \overline{h_\psi(\varphi_t(x, v))} dt \right) \mu \, d\Sigma^2 \quad (9) \\ &= \int_{\partial_+ SM} \left(\int_0^{\tau(x, v)} f(\varphi_t(x, v)) \overline{U_a^{-1}(\varphi_t(x, v)) h_\psi(\varphi_t(x, v))} dt \right) \mu \, d\Sigma^2 \\ &= \int_{SM} \left(f(x, v) \overline{U_a^{-1}(x, v) h_\psi(x, v)} \right) d\Sigma^3(x, v). \end{aligned}$$

Therefore, we obtain

$$I_a^* h = \overline{U_a^{-1} h_\psi} = U_{-\bar{a}} h_\psi. \quad (10)$$

Moreover, if $\iota_k : H_k \rightarrow L^2(SM, \mathbb{C})$ denotes the usual inclusion map, the one can show that

$$I_{m,a}^*(h) = (\overline{U_a^{-1} h_\psi})_m = (U_{-\bar{a}} h_\psi)_m,$$

where $I_{m,a} := I_a \circ \iota_k$.

From (9), one also can get the following explicit expressions for the adjoints of $I_a^0 : L^2(M, \mathbb{C}) \rightarrow L_\mu^2(\partial_+ SM, \mathbb{C})$ and $I_a^1 : L^2(\Lambda^1(M), \mathbb{C}) \rightarrow L_\mu^2(\partial_+ SM, \mathbb{C})$:

$$\begin{aligned} (I_a^0)^*(h)(x) &= \int_{S_x M} U_{-\bar{a}}(x, v) h_\psi(x, v) d\sigma_x(v), \\ (I_a^1)^*(h)^i(x) &= \left(\int_{S_x M} v^i U_{-\bar{a}}(x, v) h_\psi(x, v) d\sigma_x(v) \right), \end{aligned} \quad (11)$$

where $d\sigma_x$ is the measure on $S_x M$. The identifications of H_0 with $L^2(M, \mathbb{C})$ and $H_{-1} \oplus H_1$ with $L^2(\Lambda^1(M), \mathbb{C})$ imply that

$$(I_a^0)^*(h) = 2\pi(U_{-\bar{a}} h_\psi)_0, \quad (12)$$

$$(I_a^1)^*(h) = \pi(U_{-\bar{a}} h_\psi)_{-1} + \pi(U_{-\bar{a}} h_\psi)_1, \quad (13)$$

see [22, Remark 5.2] for more details. Then we have

$$\mathcal{I}_a^* h = [(I_a^1)^*(h), (I_a^0)^*(h)].$$

Remark 4.1. Note that $\text{Im } \mathcal{I}_a^*$ is in the orthogonal complement to $\ker \mathcal{I}_a$, and hence if $\ker \mathcal{I}_a = \mathcal{L}_{a,\text{pot}}^2(M, \mathbb{C})$ then

$$\text{Im } \mathcal{I}_a^* \subset \mathcal{L}_{a,\text{sol}}^2(M, \mathbb{C}).$$

4.2. Surjectivity of \mathcal{I}_a^* . The aim of this section is to prove the following result which is the analogue of the corresponding results in [1, 5, 22, 27, 26].

Theorem 4.2. Let (M, g) be a simple surface and let $a \in C^\infty(M, \mathbb{C})$. Suppose that $[\alpha, f] \in \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$. Then there is $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ such that $\mathcal{I}_a^*(w) = [\alpha, f]$.

Applying Theorem 4.2 to a pair of the form $[\alpha, 0]$ yields the following

Corollary 4.3. Let (M, g) be a simple surface and let $a \in C^\infty(M, \mathbb{C})$. Suppose that $\alpha : TM \rightarrow \mathbb{C}$ is a smooth, solenoidal one-form. Then there is $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ such that $(I_a^1)^*(w) = \alpha$ and $(I_a^0)^*(w) = 0$.

Proof of Theorem 4.2. We embed M into the interior of a compact surface \widetilde{M} with boundary and extend the metric g to \widetilde{M} and keep the same notation for the extension, choosing (\widetilde{M}, g) to be sufficiently close to (M, g) so that it remains simple. We also extend the attenuation coefficient a to \widetilde{M} smoothly and to be real valued, and keep the same notation for the extensions.

Before proceeding further let us introduce more conventions which will be used in this section. If A is a notation for some object in the context of the surface (M, g) , then by \widetilde{A} we denote the same object but in the context of the extended surface (\widetilde{M}, g) . For example, the notation $\widetilde{\mathcal{I}}_a$ will denote the ray transform with domain $\mathcal{L}^2(\widetilde{M}, \mathbb{C})$ and $\widetilde{\mathcal{N}}_a := (\widetilde{\mathcal{I}}_a)^* \mathcal{I}_a$.

According to Remark 4.1, we have

$$\delta_a \widetilde{\mathcal{N}}_a = 0. \quad (14)$$

Let r_M denote the restriction operator from \widetilde{M} to M . We show the following:

Lemma 4.4. The operator

$$r_M \widetilde{\mathcal{N}}_a : \mathcal{C}_0^\infty(\widetilde{M}^{\text{int}}, \mathbb{C}) \rightarrow \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C}) \quad (15)$$

is surjective.

Assuming this result, we give the proof of Theorem 4.2. Suppose that $[\alpha, f] \in \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$. Then Lemma 4.4 ensure the existence of $[\beta, h] \in \mathcal{C}_0^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})$ such that

$$[\alpha, f] = r_M \widetilde{\mathcal{N}}_a[\beta, h] = r_M (\widetilde{\mathcal{I}}_a)^* \widetilde{\mathcal{I}}_a[\beta, h].$$

Recall that \widetilde{U}_a is the unique solution to the initial value problem

$$(X + a)\widetilde{U}_a = 0 \text{ in } S\widetilde{M}, \quad \widetilde{U}_a|_{\partial_+ S\widetilde{M}} = 1.$$

The integral expression for \widetilde{U}_a is as follows:

$$\widetilde{U}_a(x, v) = \exp \left(- \int_{-\bar{\tau}(x, -v)}^0 a(\gamma_{x,v}(s)) ds \right), \quad (x, v) \in S\widetilde{M}.$$

Here $\tilde{\tau}(x, v)$ is the unique non-negative time when the geodesic $\gamma_{x,v}$, with $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$, exits \widetilde{M} . Let us now define

$$\tilde{w}(x, v) := \int_{-\tilde{\tau}(x, -v)}^{\tilde{\tau}(x, v)} \tilde{U}_a^{-1}(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) \{ \beta_j(\gamma_{x,v}(t)) \dot{\gamma}_{x,v}^j(t) + h(\gamma_{x,v}(t)) \} dt.$$

Note that $\tilde{w} \in C^\infty(S\widetilde{M}^{\text{int}}, \mathbb{C})$. From the definition, one can show that

$$\tilde{\mathcal{I}}_a[\beta, h] = \tilde{w}|_{\partial_+ S\widetilde{M}}.$$

Using the formulas (12) and (13) for the adjoints and using that $\tilde{\mathcal{I}}_a = \tilde{I}_a^1 + \tilde{I}_a^0$, we can obtain

$$\begin{aligned} [\alpha, f] &= r_M(\tilde{\mathcal{I}}_a)^* \tilde{\mathcal{I}}_a[\beta, h] \\ &= (\pi(U_{-\bar{a}}\tilde{w})_{-1} + 2\pi(U_{-\bar{a}}\tilde{w})_0 + \pi(U_{-\bar{a}}\tilde{w})_1)|_{SM} \\ &= (\mathcal{I}_a)^*(\tilde{U}_{-\bar{a}}\tilde{w})|_{\partial_+ SM}. \end{aligned}$$

In the last step we used the fact that $\tilde{U}_{-\bar{a}}$ and $U_{-\bar{a}}$ are related by

$$\tilde{U}_{-\bar{a}}(x, v) = U_{-\bar{a}}(x, v)(\tilde{U}_{-\bar{a}}|_{\partial_+ SM})_\psi(x, v) \text{ for all } (x, v) \in SM.$$

This is easy to see from the integral expressions for $\tilde{U}_{-\bar{a}}$ and $U_{-\bar{a}}$. Setting $w = (\tilde{U}_{-\bar{a}}\tilde{w})|_{\partial_+ SM}$ we finish the proof. \square

To prove Lemma 4.4 we need the following result.

Lemma 4.5. *The normal operator $\tilde{\mathcal{N}}_a$ is a pseudodifferential operator of order -1 in $\widetilde{M}^{\text{int}}$. Moreover, $\tilde{\mathcal{N}}_a$ is elliptic on a -solenoidal pairs.*

We say that $\tilde{\mathcal{N}}_a$ is *elliptic on a -solenoidal pairs*, if $\text{diag}(d_a \Lambda \delta_a, \tilde{\mathcal{N}}_a)$, acting on pairs, is elliptic (as a system of pseudodifferential operators of order -1), where Λ is a proper pseudodifferential operator on $\widetilde{M}^{\text{int}}$ with principal symbol $1/|\xi|^3$. Recall that $\text{diag}(d_a \Lambda \delta_a, \tilde{\mathcal{N}}_a)$ is an elliptic system if $\det \sigma_p(\text{diag}(d_a \Lambda \delta_a, \tilde{\mathcal{N}}_a))(x, \xi) \neq 0$ for $(x, \xi) \in T\widetilde{M} \setminus \{0\}$; see [36, page 46].

Proof. First, we prove that $\tilde{\mathcal{N}}_a$ is a pseudodifferential operator of order -1 in $\widetilde{M}^{\text{int}}$. Recall that by \tilde{U}_a we denote the unique solution to

$$(X + a)\tilde{U}_a = 0 \quad (S\widetilde{M}), \quad \tilde{U}_a|_{\partial_+ S\widetilde{M}} = 1.$$

Recall also that the normal operator is as follows $\tilde{\mathcal{N}}_a : \mathcal{L}^2(\widetilde{M}, \mathbb{C}) \rightarrow \mathcal{L}^2(\widetilde{M}, \mathbb{C})$. Therefore, we introduce the following notation

$$\tilde{\mathcal{N}}_a[\alpha, f] = [\tilde{\mathcal{N}}_a^{11}\alpha + \tilde{\mathcal{N}}_a^{10}f, \tilde{\mathcal{N}}_a^{01}\alpha + \tilde{\mathcal{N}}_a^{00}f],$$

with

$$\tilde{\mathcal{N}}_a^{11} := (\tilde{I}_a^1)^* \tilde{I}_a^1, \quad \tilde{\mathcal{N}}_a^{10} := (\tilde{I}_a^1)^* \tilde{I}_a^0, \quad \tilde{\mathcal{N}}_a^{01} := (\tilde{I}_a^0)^* \tilde{I}_a^1, \quad \tilde{\mathcal{N}}_a^{00} := (\tilde{I}_a^0)^* \tilde{I}_a^0.$$

Using (11), one can show that

$$\begin{aligned} \left(\tilde{\mathcal{N}}_a^{11} \alpha \right)^{i'}(x) &= \int_{S_x \widetilde{M}} v^{i'} \tilde{U}_{-\bar{a}}(x, v) \int_{-\tilde{\tau}(x, -v)}^{\tilde{\tau}(x, v)} \tilde{U}_a^{-1}(\varphi_{x, v}(t)) \alpha_i(\gamma_{x, v}(t)) \dot{\gamma}_{x, v}^i(t) dt d\sigma_x(v), \\ \left(\tilde{\mathcal{N}}_a^{10} f \right)^{i'}(x) &= \int_{S_x \widetilde{M}} v^{i'} \tilde{U}_{-\bar{a}}(x, v) \int_{-\tilde{\tau}(x, -v)}^{\tilde{\tau}(x, v)} \tilde{U}_a^{-1}(\varphi_{x, v}(t)) f(\gamma_{x, v}(t)) dt d\sigma_x(v), \\ \left(\tilde{\mathcal{N}}_a^{01} \alpha \right)^i(x) &= \int_{S_x \widetilde{M}} \tilde{U}_{-\bar{a}}(x, v) \int_{-\tilde{\tau}(x, -v)}^{\tilde{\tau}(x, v)} \tilde{U}_a^{-1}(\varphi_{x, v}(t)) \alpha_i(\gamma_{x, v}(t)) \dot{\gamma}_{x, v}^i(t) dt d\sigma_x(v), \\ \left(\tilde{\mathcal{N}}_a^{00} f \right)^i(x) &= \int_{S_x \widetilde{M}} \tilde{U}_{-\bar{a}}(x, v) \int_{-\tilde{\tau}(x, -v)}^{\tilde{\tau}(x, v)} \tilde{U}_a^{-1}(\varphi_{x, v}(t)) f(\gamma_{x, v}(t)) dt d\sigma_x(v). \end{aligned}$$

Following [9, 11], we use [4, Lemma B.1] to deduce that $\tilde{\mathcal{N}}_a$ is a pseudodifferential operator of order -1 , and the principal symbols of the above operators are as follows:

$$\begin{aligned} \sigma_p(\tilde{\mathcal{N}}_a^{11})^{i'i}(x, \xi) &= 2\pi \int_{S_x \widetilde{M}} \omega^{i'} \omega^i \delta(\langle \omega, \xi \rangle_{g(x)}) \tilde{U}_{-2\operatorname{Re}(a)}(x, \omega) d\sigma_x(\omega), \\ \sigma_p(\tilde{\mathcal{N}}_a^{10})^{i'}(x, \xi) &= 2\pi \int_{S_x \widetilde{M}} \omega^{i'} \delta(\langle \omega, \xi \rangle_{g(x)}) \tilde{U}_{-2\operatorname{Re}(a)}(x, \omega) d\sigma_x(\omega), \\ \sigma_p(\tilde{\mathcal{N}}_a^{01})^i(x, \xi) &= 2\pi \int_{S_x \widetilde{M}} \omega^i \delta(\langle \omega, \xi \rangle_{g(x)}) \tilde{U}_{-2\operatorname{Re}(a)}(x, \omega) d\sigma_x(\omega), \\ \sigma_p(\tilde{\mathcal{N}}_a^{00})^i(x, \xi) &= 2\pi \int_{S_x \widetilde{M}} \delta(\langle \omega, \xi \rangle_{g(x)}) \tilde{U}_{-2\operatorname{Re}(a)}(x, \omega) d\sigma_x(\omega). \end{aligned}$$

Now, we prove ellipticity. For this, note that the ellipticity of $\operatorname{diag}(d_a \Lambda \delta_a, \tilde{\mathcal{N}}_a)$ is equivalent to saying that the principal symbol $\sigma_p(\operatorname{diag}(d_a \Lambda \delta_a, \tilde{\mathcal{N}}_a))(x, \xi)$, acting on pairs, is injective for every $(x, \xi) \in T\widetilde{M} \setminus \{0\}$; see the comments preceding [40, Definition 7.1]. Assume that $\sigma_p(\tilde{\mathcal{N}}_a)[\alpha, f] = 0$ and $\sigma_p(d_a \Lambda \delta_a)[\alpha, f] = 0$ for some x and $\xi \neq 0$. Then it follows that

$$\alpha_i(x) \xi^i = 0 \tag{16}$$

and

$$\begin{aligned} (\sigma_p(\tilde{\mathcal{N}}_a)[\alpha, f], [\alpha, f])_g &= 2\pi \int_{S_x \widetilde{M}} |\alpha_i(x) \omega^i + f(x)|^2 \delta(\langle \omega, \xi \rangle_{g(x)}) \tilde{U}_{-2\operatorname{Re}(a)}(x, \omega) d\sigma_x(\omega) \\ &= 0, \end{aligned}$$

where the inner product $(\cdot, \cdot)_g$ is as in (5) before the integration. Note that $U_{-2\operatorname{Re}(a)} > 0$ and that the set $S_{x, \xi} := \{\omega \in S_x \widetilde{M} : \langle \omega, \xi \rangle_{g(x)} = 0\}$ is non-empty. Therefore, for all such ω , we get

$$\alpha_i(x) \omega^i + f(x) = 0. \tag{17}$$

Since $-\omega$ is also in $S_{x, \xi}$, we have

$$-\alpha_i(x) \omega^i + f(x) = 0.$$

These two equalities imply that $f(x) = 0$. Then (16) and (17) show that $\alpha(x) = 0$. Thus, $\tilde{\mathcal{N}}_a$ is elliptic on a -solenoidal pairs. \square

Now we give the proof of Lemma 4.4.

Proof of Lemma 4.4. For the proof we closely follow the arguments in [5]. First, we will show that $r_M \tilde{\mathcal{N}}_a$ has closed range. Since $\text{diag}(\tilde{\mathcal{N}}_a, d_a \Lambda \delta_a)$, acting on pairs, is elliptic, there is a parametrix

$$P = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$$

such that

$$\text{diag}(\tilde{\mathcal{N}}_a, d_a \Lambda \delta_a) P = \begin{pmatrix} \tilde{\mathcal{N}}_a X & \tilde{\mathcal{N}}_a Y \\ d_a \Lambda \delta_a Z & d_a \Lambda \delta_a T \end{pmatrix} \equiv \text{Id}, \quad (18)$$

and

$$P \text{diag}(\tilde{\mathcal{N}}_a, d_a \Lambda \delta_a) = \begin{pmatrix} X \tilde{\mathcal{N}}_a & Y d_a \Lambda \delta_a \\ Z \tilde{\mathcal{N}}_a & T d_a \Lambda \delta_a \end{pmatrix} \equiv \text{Id}, \quad (19)$$

where \equiv means equivalence up to a smoothing operator.

Let us use the convention that for two pairs of operators the multiplication is defined as

$$(A, B)(C, D) = AC + BD = (A, 0)(C, 0) + (0, B)(0, D).$$

If we denote $C_a := (\tilde{\mathcal{N}}_a, d_a \Lambda \delta_a)$, then from (18) and (19) there is a pair of pseudo-differential operators (A, B) such that

$$\begin{aligned} (A, B)C_a &= (A, 0)(\tilde{\mathcal{N}}_a, 0) + (0, B)(0, d_a \Lambda \delta_a) \equiv \text{Id}, \\ C_a(A, B) &= (\tilde{\mathcal{N}}_a, 0)(A, 0) + (0, d_a \Lambda \delta_a)(0, B) \equiv \text{Id}. \end{aligned} \quad (20)$$

In fact, $A = \frac{1}{2}X$ and $B = \frac{1}{2}T$. Using (14), we show that

$$-\delta_a C_a = (-\delta_a \tilde{\mathcal{N}}_a, -\delta_a d_a \Lambda \delta_a) = (0, -\delta_a d_a \Lambda \delta_a) = -\delta_a d_a (0, \Lambda \delta_a).$$

The operator $-\delta_a d_a$ is $-\Delta_g + |a|^2$, which has a proper parametrix $(-\Delta_g + |a|^2)^{-1}$. Then

$$(0, \Lambda \delta_a) = -(-\Delta_g + |a|^2)^{-1} \delta_a C_a.$$

Therefore,

$$(\tilde{\mathcal{N}}_a, 0)(A, 0) - d_a (-\Delta_g + |a|^2)^{-1} \delta_a C_a (0, B) \equiv \text{Id}.$$

Since $C_a(0, B) = C_a(A, B) - (\tilde{\mathcal{N}}_a, 0)(A, 0) \equiv \text{Id} - (\tilde{\mathcal{N}}_a, 0)(A, 0)$ and $\delta_a \tilde{\mathcal{N}}_a = 0$, this imply that

$$\tilde{\mathcal{N}}_a A - d_a (-\Delta_g + |a|^2)^{-1} \delta_a = \text{Id} + K,$$

where K is a smoothing operator in $\widetilde{M}^{\text{int}}$. Restricting to $\mathcal{C}_{0,a,\text{sol}}^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})$, we obtain

$$\tilde{\mathcal{N}}_a A[\alpha, f] = [\alpha, f] + K[\alpha, f], \quad \text{for all } [\alpha, f] \in \mathcal{C}_{0,a,\text{sol}}^\infty(\widetilde{M}^{\text{int}}, \mathbb{C}).$$

By Proposition 3.3, there is a precompact neighborhood W of M in $\widetilde{M}^{\text{int}}$ such that there is a bounded map $\mathcal{E}_a : \mathcal{H}_{a,\text{sol}}^1(M, \mathbb{C}) \rightarrow \mathcal{H}_{W,a,\text{sol}}^1(\widetilde{M}^{\text{int}}, \mathbb{C})$ such that $\mathcal{E}_a|_M = \text{Id}$ and $\mathcal{E}_a(\mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})) \subset \mathcal{C}_{W,a,\text{sol}}^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})$. Then we have on $\mathcal{H}_{a,\text{sol}}^1(M, \mathbb{C})$

$$r_M \tilde{\mathcal{N}}_a A \mathcal{E}_a = \text{Id} + r_M K \mathcal{E}_a.$$

Since K is smoothing in $\widetilde{M}^{\text{int}}$, the operator $r_M K \mathcal{E}_a$ is compact. Hence, the operator $\text{Id} + r_M K \mathcal{E}_a : \mathcal{H}_{a,\text{sol}}^1(M, \mathbb{C}) \rightarrow \mathcal{H}_{a,\text{sol}}^1(M, \mathbb{C})$ has closed and finite codimensional range. Since K is smoothing, we also get that the operator $\text{Id} + r_M K \mathcal{E}_a : \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C}) \rightarrow \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$ also has closed and finite codimensional range. Therefore, $r_M \widetilde{\mathcal{N}}_a A \mathcal{E}_a(\mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C}))$ is closed and has finite codimension in $\mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$. Since

$$r_M \widetilde{\mathcal{N}}_a A \mathcal{E}_a(\mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})) \subset r_M \widetilde{\mathcal{N}}_a(\mathcal{C}_0^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})) \subset \mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C}),$$

the intermediate space $r_M \widetilde{\mathcal{N}}_a(\mathcal{C}_0^\infty(\widetilde{M}^{\text{int}}, \mathbb{C}))$ is also closed in $\mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$.

Next, we show that the adjoint operator $(r_M \widetilde{\mathcal{N}}_a)^*$ has trivial kernel. According to (6), each functional on $\mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$ gives rise to a functional on $\mathcal{C}^\infty(M, \mathbb{C})$ that vanishes on $\mathcal{C}_{a,\text{pot}}^\infty(M, \mathbb{C})$. Therefore, the dual of $\mathcal{C}_{a,\text{sol}}^\infty(M, \mathbb{C})$ is

$$\begin{aligned} \mathcal{D}'_{M,\delta_a}(\widetilde{M}^{\text{int}}, \mathbb{C}) \\ = \{[\alpha, f] \in \mathcal{D}'(\widetilde{M}^{\text{int}}, \mathbb{C}) : \text{supp}[\alpha, f] \subset M, \langle [\alpha, f] | [\tilde{\beta}, \tilde{h}] \rangle = 0, \forall [\beta, h] \in \mathcal{C}_{a,\text{pot}}^\infty(M, \mathbb{C})\}, \end{aligned}$$

where $[\tilde{\beta}, \tilde{h}] \in \mathcal{C}^\infty(\widetilde{M}^{\text{int}})$ is any extension of $[\beta, h]$ from M to $\widetilde{M}^{\text{int}}$. Then dual operator of (15) is

$$(r_M \widetilde{\mathcal{N}}_a)^* : \mathcal{D}'_{M,\delta_a}(\widetilde{M}^{\text{int}}, \mathbb{C}) \rightarrow \mathcal{D}'(\widetilde{M}^{\text{int}}). \quad (21)$$

For all $[\alpha, f] \in \mathcal{D}'_{M,\delta_a}(\widetilde{M}^{\text{int}}, \mathbb{C})$ and $[\beta, h] \in \mathcal{C}^\infty(\widetilde{M}^{\text{int}})$

$$\langle (r_M \widetilde{\mathcal{N}}_a)^*[\alpha, f] | [\beta, h] \rangle = \langle [\alpha, f] | (r_M \widetilde{\mathcal{N}}_a[\beta, h])^\sim \rangle = \langle [\alpha, f] | \widetilde{\mathcal{N}}_a[\beta, h] \rangle = \langle \widetilde{\mathcal{N}}_a[\alpha, f] | [\beta, h] \rangle.$$

Hence,

$$(r_M \widetilde{\mathcal{N}}_a)^* = \widetilde{\mathcal{N}}_a|_{\mathcal{D}'_{M,\delta_a}(\widetilde{M}^{\text{int}}, \mathbb{C})}.$$

Suppose now that $[\alpha, f] \in \mathcal{D}'_{M,\delta_a}(\widetilde{M}^{\text{int}}, \mathbb{C})$ is in the kernel of $\widetilde{\mathcal{N}}_a$. Then from the definition of the space $\mathcal{D}'_{M,\delta_a}(\widetilde{M}^{\text{int}}, \mathbb{C})$ it follows that

$$\text{sing supp } \delta_a[\alpha, f] \subset \partial M. \quad (22)$$

Decomposing $[\alpha, f] = [\beta, h] + d_a b$ with $\delta_a[\beta, h] = 0$, we have

$$\delta_a d_a b = \delta_a[\alpha, f].$$

Since $-\delta_a d_a = -\Delta_g + |a|^2$ is an elliptic operator, (22) implies

$$\text{sing supp } b \subset \partial M. \quad (23)$$

Since $[\alpha, f]$ is supported in M , from the decomposition $[\alpha, f] = [\beta, h] + d_a b$ and (23) we say

$$\text{sing supp } [\beta, h] \subset M. \quad (24)$$

Now consider a smooth function p on \widetilde{M} equal to b in a neighborhood of $\partial \widetilde{M}$. Then

$$\widetilde{\mathcal{N}}_a d_a p = \widetilde{\mathcal{N}}_a d_a b,$$

and hence

$$\widetilde{\mathcal{N}}_a[\beta, h] = \widetilde{\mathcal{N}}_a[\alpha, f] - \widetilde{\mathcal{N}}_a d_a b = -\widetilde{\mathcal{N}}_a d_a p. \quad (25)$$

This implies that $\tilde{\mathcal{N}}_a[\beta, h]$ is smooth in $\widetilde{M}^{\text{int}}$. Now using the fact that $\delta_a[\beta, h] = 0$ and (20), we obtain that $[\beta, h]$ is smooth in $\widetilde{M}^{\text{int}}$ and hence according to (24), we conclude that $[\beta, h]$ is smooth on \widetilde{M} .

By (25), we have $\tilde{\mathcal{N}}_a([\beta, h] + d_a p) = 0$ with $[\beta, h] \in C^\infty(\widetilde{M}, \mathbb{C})$ and $p \in C^\infty(\widetilde{M}, \mathbb{C})$. Then $\tilde{\mathcal{I}}_a([\beta, h] + d_a p) = 0$ and hence, by the injectivity result [32, Theorem 1.2], $[\beta, h] + d_a p = d_a q$ for some $q \in C^\infty(\widetilde{M}, \mathbb{C})$ with $q|_{\partial \widetilde{M}} = 0$. This, combined with the decomposition $[\alpha, f] = [\beta, h] + d_a b$, gives

$$[\alpha, f] = -d_a p + d_a q + d_a b.$$

Therefore, for every $[\gamma, v] \in \mathcal{C}_{a, \text{sol}}^\infty(M, \mathbb{C})$ we have

$$\langle [\alpha, f] | [\tilde{\gamma}, \tilde{v}] \rangle = \langle [d_a(-p + q + b)] | [\tilde{\gamma}, \tilde{v}] \rangle = -\langle (-p + q + b) | \delta_a[\tilde{\gamma}, \tilde{v}] \rangle,$$

where $[\tilde{\gamma}, \tilde{v}] \in \mathcal{C}_0^\infty(\widetilde{M}^{\text{int}}, \mathbb{C})$ is any extension of $[\gamma, v]$. By Proposition 3.3, we can take $[\tilde{\gamma}, \tilde{v}]$ to satisfy $\delta_a[\tilde{\gamma}, \tilde{v}] = 0$. Therefore, $[\alpha, f]$ annihilates $\mathcal{C}_{a, \text{sol}}^\infty(M, \mathbb{C})$. By the definition of $\mathcal{D}'_{M, \delta_a}(\widetilde{M}^{\text{int}}, \mathbb{C})$ we then have $[\alpha, f] = 0$. \square

5. PROOFS OF THEOREMS 2.1 AND 2.2

Following [26, 22], we start with deriving the appropriate factorization for the operator P_a . Suppose $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$. Then w^\sharp is a smooth solution of the transport equation $(X + a)w^\sharp = 0$. Applying commutator formula (4) to w^\sharp , we obtain

$$-(X + a)Hw^\sharp = X_\perp w_0^\sharp + (X_\perp w^\sharp)_0.$$

Note that $X_\perp w_0^\sharp = \star dw_0^\sharp$. Since $X_\perp = i(\eta_- - \eta_+)$, using [22, Lemma 6.2], we also have

$$(X_\perp w^\sharp)_0 = i(\eta_- w_1^\sharp - \eta_+ w_{-1}^\sharp) = \frac{1}{2} \star d(w_{-1}^\sharp + w_1^\sharp).$$

Therefore,

$$-2\pi(X + a)Hw^\sharp = 2\pi \star dw_0^\sharp + \pi \star d(w_{-1}^\sharp + w_1^\sharp).$$

Applying I_a to the above equality and using the expressions for the adjoint of the ray transform in (12) and (13), we deduce

$$-2\pi P_a w = \mathcal{I}_a[\star d(I_{-\bar{a}}^0)^*(w), \star d(I_{-\bar{a}}^1)^*(w)] = \mathcal{I}_a \begin{bmatrix} 0 & \star d \\ \star d & 0 \end{bmatrix} \mathcal{I}_{-\bar{a}}^* w. \quad (26)$$

Proof of Theorem 2.2. Proof of Claim (1). Suppose that $u = P_a w$ for some $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ with $w_0^\sharp = 0$. According to (12), $w_0^\sharp = 0$ is equivalent to saying $(I_{-a}^0)^*(w) = 0$, the factorization (26) shows that u belongs to the range of I_a^0 .

Conversely, suppose $u = I_a^0 f$ for some $f \in C^\infty(M, \mathbb{C})$. By basic properties of the Hodge star \star , we know that $f = \star(f d \text{Vol}_g)$. Since M is simply connected and $f d \text{Vol}_g$ is closed, there is a smooth one-form α such that $d\alpha = f d \text{Vol}_g$. Recall that α can be written as $\alpha = \alpha^s + dh$ where α^s is solenoidal and $h \in C^\infty(M, \mathbb{C})$ such that $h|_{\partial M} = 0$. Then $d\alpha = d\alpha^s$, since $d^2 = 0$. Therefore, without loss of generality, we can assume α to be solenoidal. Thus, we have $u = I_a^0 \star d\alpha$ with α being

solenoidal. By Corollary 4.3 there is $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ such that $(I_{-\bar{a}}^1)^*(w) = \alpha$ and $(I_{-\bar{a}}^0)^*(w) = 0$. Using (26), we can conclude that

$$u = I_a^0 \star d(I_{-\bar{a}}^1)^*(w) = \mathcal{I}_a[\star d(I_{-\bar{a}}^0)^*(w), \star d(I_{-\bar{a}}^1)^*(w)] = P_a w,$$

which finishes the proof of Claim (1).

Proof of Claim (2). Suppose that $u = P_a w$ for some $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ such that $w_{-1}^\sharp + w_1^\sharp = dp$ for some $p \in C^\infty(M, \mathbb{C})$. According to (13), $w_{-1}^\sharp + w_1^\sharp = dp$ is equivalent to saying $(I_{-\bar{a}}^1)^*(w) = dq$ for some $q \in C^\infty(M, \mathbb{C})$. Then the factorization (26) shows that u belongs to the range of I_a^1 acting on solenoidal one-forms. Conversely, suppose $u = I_a^1 \star d\varphi$ for some $\varphi \in C^\infty(M, \mathbb{C})$. Since the Hodge star operator \star is isomorphism between $\Omega^2(M, \mathbb{C})$ and $C^\infty(M, \mathbb{C})$, there is a two-form ω such that $\star\omega = a\varphi$. Since M is simply connected and ω is closed, there is a smooth one-form β such that $\omega = d\beta$. As in the proof of Claim 1, β can be taken to be solenoidal, i.e. $\beta = \star dh$. Write $\alpha = \star\beta = -dh$, then one can check that $\delta_{-\bar{a}}[\alpha, \varphi] = 0$. Then by Theorem 4.2 there is $w \in \mathcal{S}_a^\infty(\partial_+ SM, \mathbb{C})$ such that $\mathcal{I}_{-\bar{a}}^*(w) = [\alpha, \varphi]$. Since $d\alpha = 0$, using (26), we can conclude that

$$u = I_a^1 \star d\varphi = \mathcal{I}_a[\star d(I_{-\bar{a}}^0)^*(w), \star d(I_{-\bar{a}}^1)^*(w)] = P_a w.$$

According to (13), since $\alpha = -dh$, we have

$$w_{-1}^\sharp + w_1^\sharp = \frac{1}{\pi}(I_{-\bar{a}}^1)^*(w) = dq, \quad q = -\frac{1}{\pi}h.$$

Hence, the proof of Claim (2) is complete. \square

6. AN INJECTIVE DECOMPOSITION OF THE RANGE OF \mathcal{I}_a

While spaces of pairs are more amenable to the microlocal analysis arguments from the previous sections, inverting \mathcal{I}_a over pairs requires finding a representative modulo the kernel of \mathcal{I}_a of a -potential pairs. One way to achieve this below is to use a different domain of definition, over which the transform is *injective*. We first define the mapping

$$\dot{C}^\infty(M) \times C^\infty(M) \ni (h, f) \mapsto \mathcal{I}_a[\star dh, f] \in C^\infty(\partial_+ SM),$$

where we have defined the space

$$\dot{C}^\infty(M) := \left\{ h \in C^\infty(M) : \int_{\partial M} h(s) ds = 0 \right\}, \quad (27)$$

(note that any other normalization condition setting constants to zero may work) and we establish the following:

Theorem 6.1. *Let (M, g) a simple surface with boundary and $a \in C^\infty(M)$. Then:*

- (i) *The transform $\dot{C}^\infty(M) \times C^\infty(M) \ni (h, f) \mapsto \mathcal{I}_a[\star dh, f]$ is injective.*
- (ii) *For any smooth pair $[\alpha, b] \in \mathcal{C}^\infty(M, \mathbb{C})$, there exists a unique couple $(h, f) \in \dot{C}^\infty(M) \times C^\infty(M)$ such that $\mathcal{I}_a[\alpha, b] = \mathcal{I}_a[\star dh, f]$.*

Proof of Theorem 6.1. Proof of (i). Suppose that (h, f) are such that $\mathcal{I}_a[\star dh, f] = 0$. By solenoidal injectivity of \mathcal{I}_a , this implies that $[\star dh, f] = d_a m = [dm, am]$ for some function m vanishing on ∂M . Then the equality $\star dh = dm$ implies that m and h are harmonic. Since $m|_{\partial M} = 0$, then $m = 0$ on M . In turn, $f = am = 0$ and since $\star dh = 0$ h is constant equal to zero, due to the normalization condition (27).

Proof of (ii). Let $[\alpha, b]$ a smooth pair. Then $\mathcal{I}_a[\alpha, b] = u|_{\partial_+ SM}$, where u is the solution to

$$Xu + au = -b - \alpha(v) \quad (SM), \quad u|_{\partial_- SM}.$$

Now, α has a unique Hodge decomposition $\alpha = df' + \star dh$ with $f' \in C^\infty(M)$ with $f'|_{\partial M} = 0$ and $h \in \dot{C}^\infty(M)$. As functions on SM , this means, $\alpha(v) = Xf' + X_\perp h$, and thus the previous transport equation can be rewritten as

$$X(u + f') + a(u + f') = -(b - af') - X_\perp h,$$

where the functions $u + f'$ and u agree on $\partial_\pm SM$. In particular, $(u + f')|_{\partial_+ SM} = 0$ and

$$\mathcal{I}_a[\alpha, b] = u|_{\partial_+ SM} = (u + f')|_{\partial_+ SM} = \mathcal{I}_a[\star dh, b - af'].$$

Therefore, the couple $(h, b - af')$ provides the desired candidate, whose smoothness comes from elliptic regularity and smoothness of a . In addition, such a couple is unique by virtue of (i). Theorem 6.1 is proved. \square

We now decompose h further. Recall that we define $\ker^k \eta_\pm := \Omega_k \cap \ker \eta_\pm$.

Lemma 6.2. *Any $h \in \dot{C}^\infty(M)$ decomposes into $h = h_0 + h_+ + h_-$, where $h_0 \in C_0^\infty(M)$ is unique and $h_\pm \in \ker^0 \eta_\pm$ are unique up to a constant. In particular, $h = 0$ if and only if $h_0 = 0$ and h_+ and h_- are constant.*

Proof. Let $h \in \dot{C}^\infty(M)$ and define u unique harmonic function with $u|_{\partial M} = h|_{\partial M}$. By elliptic regularity and smoothness of ∂M , $u \in \dot{C}^\infty(M)$. Let v the unique harmonic conjugate to u satisfying the normalization condition (27), such that $du = \star dv$. In the sense of functions on SM , this is equivalent to saying $Xu = X_\perp v$ which upon using that $X = \eta_+ + \eta_-$ and $X_\perp = \frac{1}{i}(\eta_+ - \eta_-)$, yields

$$\eta_+(u + iv) + \eta_-(u - iv) = 0.$$

Projecting onto Ω_1 and Ω_{-1} gives $\eta_+(u + iv) = 0$ and $\eta_-(u - iv)$. Therefore, the decomposition follows upon writing

$$h = (h - u) + \frac{1}{2}(u + iv) + \frac{1}{2}(u - iv).$$

Lemma 6.2 is proved. \square

Upon decomposing $h = h_0 + h_+ + h_-$ as in Lemma 6.2, and using that $h_\pm \in \ker^0 \eta_\pm$, the data $\mathcal{D} := \mathcal{I}_a[\star dh, f]$ looks like

$$\begin{aligned} \mathcal{D} &= I_a(f + X_\perp h_0 - i\eta_+ h_- + i\eta_- h_+) \\ &= I_a^0 f + I_a^\perp h_0 + I_a^{+1}(-i\eta_+ h_-) + I_a^{-1}(i\eta_- h_+), \end{aligned}$$

where the equality does not depend on constants added to h_+ or h_- . From the commutator relation $[\eta_+, \eta_-] = \frac{i}{2}\kappa V$, we can see that $\eta_+ h_- \in \ker^1 \eta_-$ and $\eta_- h_+ \in \ker^{-1} \eta_+$. Upon defining $\omega_1 = -i\eta_+ h_-$ and $\omega_{-1} = i\eta_- h_+$, and in light of Lemma 6.2, the decomposition $\star dh = \star dh_0 + \omega_1 + \omega_{-1}$ is unique and the left hand side is zero if and only if each summand of the right hand side is zero. Combining this with Theorem 6.1, we arrive at the following conclusion:

Lemma 6.3. *For any $\mathcal{D} \in \text{Range } \mathcal{I}_a$, there exists a unique quadruple $(f, h_0, \omega_1, \omega_{-1}) \in C^\infty(M) \times C_0^\infty(M) \times \ker^1 \eta_- \times \ker^{-1} \eta_+$ such that*

$$\mathcal{D} = I_a^0 f + I_a^1 h_0 + I_a^{+1} \omega_1 + I_a^{-1} \omega_{-1}.$$

In particular, $\mathcal{D} = 0$ if and only if the entire quadruple vanishes identically.

7. INVERSION APPROACH

As Lemma 6.3 suggests, since the mapping $(f, h_0, \omega_1, \omega_{-1}) \mapsto \mathcal{D}$ is injective, we expect to write reconstruction formulas for each element of the quadruple, which is the purpose of this section. The remainder is organized as follows:

- In Section 7.1, we will first show how to reconstruct ω_1 and ω_{-1} from \mathcal{D} , thereby allowing us to remove the data $I_a^{+1} \omega_1 + I_a^{-1} \omega_{-1}$ from \mathcal{D} .
- In Section 7.2, as a preparation toward the reconstruction of (h_0, f) , we will construct a so-called boundary holomorphization operator, related to the unattenuated transform \mathcal{I}_0 .
- In Section 7.3, we will then show how to reconstruct (h_0, f) from the remaining data $\mathcal{I}_a[\star dh_0, f]$ via explicit formulas.

7.1. Reconstruction of ω_1 and ω_{-1} . Here and below, by $\mathcal{O}_{\geq k}$ (resp. $\mathcal{O}_{\leq k}$), we denote an element $u \in C^\infty(SM)$ such that $u_p = 0$ for all $p < k$ (resp. all $p > k$). We first recall the following result from [22], see also [25].

Lemma 7.1 (Lemma 5.6 in [22]). *Given any $f \in \Omega_m$, there exists $w \in C^\infty(SM, \mathbb{C})$ such that $Xw = 0$ and $w_m = f$.*

Using Lemma 7.1, we prove the following:

Lemma 7.2. *Let (M, g) simple and $a \in C^\infty(M, \mathbb{C})$. Then the following statements hold true:*

- (1) *For any $\phi \in \ker^1 \eta_-$, there exists $w = \phi + \mathcal{O}_{\geq 2}$, solution of $Xw - \bar{a}w = 0$.*
- (2) *For any $\phi \in \ker^{-1} \eta_+$, there exists $w = \phi + \mathcal{O}_{\leq -2}$, solution of $Xw - \bar{a}w = 0$.*

Proof of Lemma 7.2. Let $\vec{w}, \overleftarrow{w}$ denote smooth, odd solutions of $X\vec{w} = X\overleftarrow{w} = \bar{a}$ with \vec{w} holomorphic and \overleftarrow{w} antiholomorphic, whose existence is established in [32, Proposition 4.1]. Then $e^{\vec{w}}$ is a holomorphic solution of $(X - \bar{a})e^{\vec{w}} = 0$ of the form $e^{\vec{w}} = 1 + \mathcal{O}_{\geq 1}$ and $e^{\overleftarrow{w}} = 1 + \mathcal{O}_{\leq -1}$ is an antiholomorphic solution of $(X - \bar{a})e^{\overleftarrow{w}} = 0$.

Proof of (1). For $\phi \in \ker^1 \eta_-$, using Lemma 7.1, there exists v smooth solution of $Xv = 0$ with $v_1 = \phi$. Since $\eta_- v_1 = \eta_- \phi = 0$, then $v' = \sum_{k \geq 1} v_k$ is another smooth solution of $Xv' = 0$ with $v'_1 = v_1 = \phi$. Then setting $w = e^{\vec{w}} v'$ completes the proof.

Proof of (2). For $\phi \in \ker^{-1} \eta_+$, using Lemma 7.1, there exists v smooth solution of $Xv = 0$ with $v_{-1} = \phi$. Since $\eta_+ v_{-1} = \eta_+ \phi = 0$, then $v' = \sum_{k \leq -1} v_k$ is another smooth solution of $Xv' = 0$ with $v'_1 = v_1 = \phi$. Then setting $w = e^{\bar{w}} v'$ completes the proof. \square

The spaces $L^2(\ker^k \eta_{\pm})$. In the sequel, we denote

$$L^2(\ker^k \eta_{\pm}) := \{f \in L^2(SM) : f_p = 0, \quad p \neq k; \eta_{\pm} f = 0\}.$$

These spaces are closed subspaces of $L^2(SM)$, essentially because, using isothermal coordinates, the operators η_{\pm} are $\partial_z, \partial_{\bar{z}}$ operators and that $L^2(M)$ -limits of solutions of $\partial_{\bar{z}} f = 0$ are in fact normal limits (uniform limits on compact subsets of M), and thus themselves solutions of $\partial_{\bar{z}} f = 0$ (see for instance [37, Ex. 6 p254]). These spaces are therefore Hilbert spaces themselves, admitting complete orthonormal sets. For the sequel, we denote $\{\phi^{\pm 1, (p)}\}_{p=0}^{\infty}$ orthonormal Hilbert bases of $L^2(\ker^{\pm 1} \eta_{\mp})$. Then for any $\phi^{1, (p)}$, we define $w^{1, (p)}$ as in Lemma 7.2.(1) and for any $\phi^{-1, (p)}$, we define $w^{-1, (p)}$ according to Lemma 7.2.(2).

We now explain how to reconstruct elements of $\ker^{\pm 1} \eta_{\mp}$ from knowledge of their ray transforms, and notice how these reconstructions pay no heed to the additional terms f and h_0 .

Theorem 7.3. *Let (M, g) a simple surface and $a \in C^{\infty}(M, \mathbb{C})$. Let $\mathcal{D} \in \text{Range } \mathcal{I}_a$ and $(f, h_0, \omega_1, \omega_{-1})$ as in Lemma 6.3, then the harmonic one-forms ω_1 and ω_{-1} can be reconstructed from $\mathcal{D} = I_a^0 f + I_a^{\perp} h_0 + I_a^{+1} \omega_1 + I_a^{-1} \omega_{-1}$ via the formulas*

$$\omega_1 = \sum_{p=0}^{\infty} \langle \mathcal{D}, w^{1, (p)} |_{\partial_+ SM} \rangle_{L^2_{\mu}(\partial_+ SM)} \phi^{1, (p)}, \quad (28)$$

$$\omega_{-1} = \sum_{p=0}^{\infty} \langle \mathcal{D}, w^{-1, (p)} |_{\partial_+ SM} \rangle_{L^2_{\mu}(\partial_+ SM)} \phi^{-1, (p)}. \quad (29)$$

Proof. We only prove (28), as the proof of (29) is similar.

Proof of (28). Recall that $\mathcal{D} = u|_{\partial_+ SM}$, where u solves the problem

$$Xu + au = -f - X_{\perp} h_0 - \omega_1 - \omega_{-1} \quad (SM).$$

For any $p \geq 0$, setting $\phi = \phi^{1, (p)}$ and $w = w^{1, (p)}$, we take the $L^2(SM)$ inner product of the transport equation above with w to make appear:

$$LHS = (Xu + au, w)_{SM} = -\langle \mathcal{D}, w |_{\partial_+ SM} \rangle_{L^2_{\mu}(\partial_+ SM)} + \overline{(u, -Xw + \bar{a}w)_{SM}}$$

$$RHS = (-f - X_{\perp} h_0 - \omega_1 - \omega_{-1}, w)_{SM} = (-\eta_+ h_0 - \omega_1, \phi)_{SM} = -(\omega_1, \phi)_{SM},$$

where the integration by parts $(\eta_+ h_0, \phi)_{SM} = (h_0, \eta_- \phi)_{SM} = 0$ holds with no boundary term since $h_0|_{\partial M} = 0$. We then arrive at the relation

$$\langle \mathcal{D}, w |_{\partial_+ SM} \rangle_{L^2_{\mu}(\partial_+ SM)} = (\omega_1, \phi)_{SM}.$$

Therefore, for $\phi = \phi^{1, (p)}$ above and $w = w^{1, (p)}$ as in Lemma 7.2.(1),

$$\langle \mathcal{D}, w^{1, (p)} |_{\partial_+ SM} \rangle_{L^2_{\mu}(\partial_+ SM)} = (\omega_1, \phi^{1, (p)})_{SM}, \quad \forall p \geq 0.$$

Since $\omega_1 \in L^2(\ker^1 \eta_-)$, then Bessel's inequality implies that

$$\sum_{p=0}^{\infty} |\langle \mathcal{D}, w^{1,(p)} |_{\partial_+ SM} \rangle_{L^2_\mu(\partial_+ SM)}|^2 = \sum_{p=0}^{\infty} |(\omega_1, \phi^{1,(p)})_{SM}|^2 \leq \|\omega_1\|_{L^2}^2,$$

so that the following infinite sum makes sense:

$$\omega_1 = \sum_{p=0}^{\infty} (\omega_1, \phi^{1,(p)})_{SM} \phi^{1,(p)} = \sum_{p=0}^{\infty} \langle \mathcal{D}, w^{1,(p)} |_{\partial_+ SM} \rangle_{L^2_\mu(\partial_+ SM)} \phi^{1,(p)},$$

hence (28) is proved. \square

Theorem 7.3 gives rise to two linear operators $L_{a,\pm 1} : \text{Range } \mathcal{I}_a \rightarrow \ker^{\pm 1} \eta_{\mp}$ satisfying

$$\begin{aligned} I_a L_{a,+1} (I_a^0 f + I_a^\perp h_0 + I_a^{+1} \omega_1 + I_a^{-1} \omega_{-1}) &= I_a^{+1} \omega_1, \\ I_a L_{a,-1} (I_a^0 f + I_a^\perp h_0 + I_a^{+1} \omega_1 + I_a^{-1} \omega_{-1}) &= I_a^{-1} \omega_{-1}. \end{aligned} \quad (30)$$

If we then define

$$P_{a,\pm 1} : \text{Range } \mathcal{I}_a \rightarrow \text{Range } \mathcal{I}_a, \quad P_{a,\pm 1} := I_a L_{a,\pm 1}, \quad (31)$$

such operators are idempotent on $\text{Range } \mathcal{I}_a$ (i.e., satisfy $P_{a,\pm 1}^2 = P_{a,\pm 1}$). In particular, applying $Id - P_{a,1} - P_{a,-1}$ to \mathcal{D} allows to remove $I_a^{+1} \omega_1$ and $I_a^{-1} \omega_{-1}$ from \mathcal{D} .

Remark 7.4. *If the data is not in the range of \mathcal{I}_a in the first place, the operators $L_{a,\pm 1}$ may pick up some additional components which are in the complement of $\text{Range } \mathcal{I}_a$. This behavior depends on the choice of first integral $w^{\pm 1,(p)}$ for $\phi^{\pm 1,(p)}$. Methods for finding such elements will be the object of future work.*

7.2. Holomorphization of solutions to unattenuated transport equations with holomorphic right-hand side. As a preparation for the reconstruction of (f, h_0) , this section focuses on the unattenuated transform

$$\dot{C}^\infty(M) \times C^\infty(M) \ni (h, f) \mapsto \mathcal{I}_0[\star dh, f],$$

in particular, how its injectivity allows to produce holomorphic solutions to transport equations with holomorphic right-hand sides, out of any other solution of the same transport problem, via a so-called boundary holomorphization operator.

For conciseness, we will denote $\mathcal{I}^{0,\perp}[h, f] = \mathcal{I}_0[\star dh, f]$, and we also denote I^0 and I^1 the unattenuated transforms over smooth functions and one-forms, and $I^\perp(h) := I^1(\star dh)$ for $h \in \dot{C}^\infty(M)$. The remarks from Section 6 imply that, while I^1 is only solenoidal-injective and has a kernel, I^\perp is injective and both transforms have the same range.

Recall the boundary operators $P_\pm := A_-^* H_\pm A_+$ defined in [26], where $A_+ w = Q_0 w = w_\psi|_{\partial SM}$ and $A_-^* = B_0$ in our current notation. One may simply define $P = A_-^* H A_+ = P_+ + P_-$, where in fact, the operators P_\pm represent the action of P on two orthogonal subspaces of $L^2_\mu(\partial_+ SM)$. In order to clarify this, let us define the *antipodal scattering relation* $\alpha_A : \partial_+ SM \rightarrow \partial_+ SM$ to be the scattering relation composed with the antipodal map $(x, v) \mapsto (x, -v)$. α_A is clearly an involution of

$\partial_+ SM$, and since the measure $\mu \, d\Sigma^2$ is preserved by the pull-back α_A^* , the following orthogonal decomposition holds

$$L_\mu^2(\partial_+ SM) = \mathcal{V}_+ \overset{\perp}{\oplus} \mathcal{V}_-, \quad \mathcal{V}_\pm := \ker(Id \mp \alpha_A^*).$$

Further inspection of symmetries upon applying the operators A_+ , then H , then A_-^* to functions in \mathcal{V}_\pm , shows that the operator P , in this decomposition, has the matrix form $P = \begin{bmatrix} 0 & P_- \\ P_+ & 0 \end{bmatrix}$. The other facts below are also obvious:

- Range $I^0 \subset \mathcal{V}_+$ thus $(I^0)^*(\mathcal{V}_-) = \{0\}$.
- Range $I^\perp \subset \mathcal{V}_-$ thus $(I^\perp)^*(\mathcal{V}_+) = \{0\}$.

Now the range characterization [26, Theorem 4.5] states that $P_- : C_\alpha^\infty(\partial_+ SM) \rightarrow C^\infty(\partial_+ SM)$ is surjective on the range of I^0 and $P_+ : C_\alpha^\infty(\partial_+ SM) \rightarrow C^\infty(\partial_+ SM)$ is surjective on the range of I^\perp . This justifies the existence of right inverses

$$\begin{aligned} P_+^\dagger : \text{Range } I^\perp &\rightarrow C_\alpha^\infty(\partial_+ SM) \cap \mathcal{V}_+, & P_+ P_+^\dagger &= Id|_{\text{Range } I^\perp}, \\ P_-^\dagger : \text{Range } I^0 &\rightarrow C_\alpha^\infty(\partial_+ SM) \cap \mathcal{V}_-, & P_- P_-^\dagger &= Id|_{\text{Range } I^0}. \end{aligned}$$

Using the factorizations $2\pi P_+ = I^\perp (I^0)^*$ and $2\pi P_- = -I^0 (I^\perp)^*$, this implies

$$\begin{aligned} 2\pi I^\perp h &= 2\pi P_+ P_+^\dagger I^\perp h = I^\perp (I^0)^* P_+^\dagger I^\perp h, \quad \forall h \in \dot{C}^\infty(M), \\ 2\pi I^0 f &= 2\pi P_- P_-^\dagger I^0 f = -I^0 (I^\perp)^* P_-^\dagger I^0 f, \quad \forall f \in C^\infty(M), \end{aligned}$$

which by injectivity of I^0 and I^\perp implies

$$\begin{aligned} \frac{1}{2\pi} (I^0)^* P_+^\dagger I^\perp h &= h + \text{constant}, \quad \forall h \in \dot{C}^\infty(M), \\ -\frac{1}{2\pi} (I^\perp)^* P_-^\dagger I^0 f &= f, \quad \forall f \in C^\infty(M). \end{aligned} \tag{32}$$

Out of the two right-inverses P_\pm^\dagger , we may construct a right inverse P^\dagger for P , defined on $\text{Range } I^0 \oplus \text{Range } I^\perp = \text{Range } \mathcal{I}^{0,\perp}$ and $C_\alpha^\infty(\partial_+ SM)$ -valued, defined by

$$P^\dagger w = P_-^\dagger \frac{1}{2} (Id + \alpha_A^*) w + P_+^\dagger \frac{1}{2} (Id - \alpha_A^*) w, \quad w \in \text{Range } \mathcal{I}^{0,\perp},$$

such that $PP^\dagger = Id$ on $\text{Range } \mathcal{I}^{0,\perp}$.

Theorem 7.5 (Holomorphization operator). *Let (M, g) a simple Riemannian surface with boundary. There exists a linear boundary operator*

$$\vec{B} : C^\infty(\partial SM) \rightarrow C^\infty(\partial_+ SM)$$

such that for any function $f \in C^\infty(SM)$ and any solution u of $Xu = -f$ smooth on SM , the function $\vec{u} := u - (\vec{B}(u|_{\partial SM}))_\psi$ satisfies:

- (1) *If $f = f_{-1} + f_0 + \sum_{k \geq 1} f_k$, then \vec{u} is holomorphic.*
- (2) *If, additionally, $f_{-1} = 0$, then \vec{u}_0 is constant.*

If P^\dagger denotes any right-inverse for P , then such an operator \vec{B} may be obtained by defining

$$\vec{B}h := \frac{1}{2} [(Id - iH)h + i(Id + iH)(A_+ P^\dagger A_-^* (Id - iH)h)]|_{\partial_+ SM}. \tag{33}$$

By complex conjugation, we state a corollary of Theorem 7.5 without proof, regarding the existence of an anti-holomorphization operator.

Corollary 7.6 (Anti-holomorphization operator). *With $\vec{\mathcal{B}}$ as in Theorem 7.5, the operator*

$$\bar{\mathcal{B}} : C^\infty(\partial SM) \rightarrow C^\infty(\partial_+ SM), \quad \bar{\mathcal{B}}h := \overline{\vec{\mathcal{B}}h}, \quad h \in C^\infty(\partial SM),$$

is such that for any function $f \in C^\infty(SM)$ and any solution u of $Xu = -f$ smooth on SM , the function $\bar{u} := u - (\vec{\mathcal{B}}(u|_{\partial SM}))_\psi$ satisfies:

- (1) If $f = \sum_{k \leq -1} f_k + f_0 + f_1$, then \bar{u} is anti-holomorphic.
- (2) If, additionally, $f_1 = 0$, then \bar{u}_0 is constant.

Remark 7.7. Using the fact that $\vec{\mathcal{B}}(0) = 0$ and $\bar{\mathcal{B}}(0) = 0$, we recover the statement of [32, Proposition 5.1]: if u solves $Xu = -f$ with f holomorphic (resp. anti-holomorphic), and $u|_{\partial SM} = 0$, then u is holomorphic (resp. antiholomorphic) and $u_0 = 0$.

Remark 7.8 (Continuity and explicitness of $\vec{\mathcal{B}}$ and $\bar{\mathcal{B}}$). *The continuity of $\vec{\mathcal{B}}$ and $\bar{\mathcal{B}}$ relies heavily on the continuity of P^\dagger , for which explicit expressions remain to be found in general. In the case where the surface is such that the operator $Id + W^2$ is invertible (see [26, 15] for a definition of W), then such a right-inverse is explicitly constructed in [15]. This is done by using the factorizations $2\pi P_+ = I^\perp(I^0)^*$ and $2\pi P_- = -I^0(I^\perp)^*$, and constructing explicit right-inverses for $(I^0)^*$, $(I^\perp)^*$, and inverting I^0 , I^\perp , using the Fredholm equations first derived in [26]. This construction is valid in the case of surfaces with Gaussian curvature close enough to constant, though whether the operator $Id + W^2$ is always invertible on simple surfaces remains open at present.*

Proof of Theorem 7.5. Let P^\dagger a right-inverse for P , let $f \in C^\infty(SM)$ and u a solution of $Xu = -f$. Write

$$u = \frac{1}{2}(u^{(+)} + u^{(-)}), \quad u^{(\pm)} := (Id \pm iH)u,$$

where $(Id - iH)u$ solves the PDE

$$X(Id - iH)u = (Id - iH)Xu + [X, Id - iH]u = -2f_{-1} - f_0 + i(X_\perp u)_0 + iX_\perp u_0.$$

Applying the Hodge decomposition to the one-form $2f_{-1}$, there exists $g \in C_0^\infty(M)$ and $h \in C^\infty(M)$ such that $2f_{-1} = Xg + X_\perp h$, in which case the previous equation can be rewritten as

$$X(u^{(-)} + g) = -(f_0 - i(X_\perp u)_0) - X_\perp(h - iu_0).$$

Upon integrating along geodesics, we make appear

$$A_-^*(u^{(-)}|_{\partial SM}) = A_-^*(u^{(-)} + g)|_{\partial SM} = I^0[f_0 - i(X_\perp u)_0] + I^\perp[h - iu_0], \quad (34)$$

where the right-hand-side belongs to $\text{Range } \mathcal{I}^{0,\perp}$.

Define $u' = -i(Id + iH)(P^\dagger A_-^*(u^{(-)}|_{\partial SM}))_\psi$. u' is holomorphic by construction and we now claim that (i) $u'_0 = -ih - u_0 + C$ (with C a constant), and (ii) u' is another solution of

$$Xu' = -(f_0 - i(X_\perp u)_0) - X_\perp(h - iu_0).$$

In both claims, we will make use of the observations that, using identities (32) and the equality (34), we have

$$\begin{aligned} \frac{1}{2\pi}(I^0)^* P^\dagger A_-^*(u^{(-)}|_{\partial SM}) &= \frac{1}{2\pi}(I^0)^* P_+^\dagger I^\perp[h - iu_0] = h - iu_0 + iC \quad (C \text{ constant}), \\ \frac{1}{2\pi}(I^\perp)^* P^\dagger A_-^*(u^{(-)}|_{\partial SM}) &= \frac{1}{2\pi}(I^\perp)^* P_-^\dagger I^0[f_0 - i(X_\perp u)_0] = -(f_0 - i(X_\perp u)_0). \end{aligned}$$

Then proving claim (i) amounts to computing

$$\begin{aligned} u'_0 &= -i \left((Id + iH)(P^\dagger A_-^*(u^{(-)}|_{\partial SM}))_\psi \right)_0 \\ &= -i \left((P^\dagger A_-^*(u^{(-)}|_{\partial SM}))_\psi \right)_0 \\ &= \frac{-i}{2\pi} (I^0)^* P^\dagger A_-^*(u^{(-)}|_{\partial SM}) \\ &= -i(h - iu_0 + iC) = -ih - u_0 + C. \end{aligned}$$

Proving claim (ii) then amounts to computing

$$\begin{aligned} Xu' &= -Xi(Id + iH)(P^\dagger A_-^*(u^{(-)}|_{\partial SM}))_\psi \\ &= XH(P^\dagger A_-^*(u^{(-)}|_{\partial SM}))_\psi \\ &= -[H, X](P^\dagger A_-^*(u^{(-)}|_{\partial SM}))_\psi \\ &= - \left(X_\perp(P^\dagger A_-^*(u^{(-)}|_{\partial SM}))_\psi \right)_0 - X_\perp \left((P^\dagger A_-^*(u^{(-)}|_{\partial SM}))_\psi \right)_0 \\ &= \frac{1}{2\pi} I_\perp^* P^\dagger A_-^*(u^{(-)}|_{\partial SM}) - \frac{1}{2\pi} X_\perp I_0^* P^\dagger A_-^*(u^{(-)}|_{\partial SM}) \\ &= -(f_0 - i(X_\perp u)_0) - X_\perp(h - iu_0 + iC). \end{aligned}$$

Now that the claims are proved, we use u' to rewrite u as

$$u = \frac{1}{2}(u^{(+)} - g + u') + \frac{1}{2}(u^{(-)} + g - u'),$$

where the first summand $\overrightarrow{u} := \frac{1}{2}(u^{(+)} - g + u')$ is holomorphic by construction, and where the second summand satisfies $X(\frac{1}{2}(u^{(-)} + g - u')) = 0$, so that it is equal to some h_ψ , where $h = \frac{1}{2}(u^{(-)} + g - u')|_{\partial SM} = \vec{B}(u|_{\partial SM})$ by construction. So Claim 1 is proved. As for Claim 2, if $f_{-1} = 0$, then the Hodge decomposition above becomes $h = g = 0$, and using claim (i), we read

$$2\overrightarrow{u}_0 = (u^{(+)} - g + u')_0 = u_0 - g - ih - u_0 + C = C.$$

Thus Theorem 7.5 is proved. \square

7.3. Reconstruction of f and h_0 . In light of Section 7.1, given the quadruple $(f, h_0, \omega_1, \omega_2)$, it is possible to extract $\mathcal{I}_a[\star dh_0, f]$ from $\mathcal{D} = I_a^0 f + I_a^\perp h_0 + I_a^{+1} \omega_1 + I_a^{-1} \omega_{-1}$ via the processing $\mathcal{I}_a[\star dh_0, f] = (Id - P_{a,1} - P_{a,-1})\mathcal{D}$. From this data, and using the results of the previous section, let us now provide explicit inversion formulas for (h_0, f) . Recall the statement of Theorem 2.3 stated in Section 2.

Theorem 7.9. *Let (M, g) a simple surface and $a \in C^\infty(M, \mathbb{C})$. Define \vec{w} and \overleftarrow{w} smooth holomorphic and antiholomorphic, odd, solutions of $X\vec{w} = X\overleftarrow{w} = -a$, and let $\vec{\mathcal{B}}$ and $\overleftarrow{\mathcal{B}}$ as in Theorem 7.5 and Corollary 7.6. Then the functions $(h_0, f) \in C_0^\infty(M) \times C^\infty(M)$ can be reconstructed from data $\mathcal{I} := \mathcal{I}_a[\star dh_0, f]$ (extended by zero on $\partial_- SM$) via the following formulas:*

$$\begin{aligned} f &= -\eta_+ \vec{\mathcal{D}}_{-1} - \eta_- \overleftarrow{\mathcal{D}}_1 - \frac{a}{2} \left(\vec{\mathcal{D}}_0 + \overleftarrow{\mathcal{D}}_0 + i(g_+ - g_-) \right), \\ h_0 &= \frac{1}{2}(g_+ + g_-) - \frac{i}{2}(\vec{\mathcal{D}}_0 - \overleftarrow{\mathcal{D}}_0), \end{aligned}$$

where we have defined $\vec{\mathcal{D}} := e^{\vec{w}}(\vec{\mathcal{B}}(\mathcal{I}e^{-\vec{w}}|_{\partial SM}))_\psi$, $\overleftarrow{\mathcal{D}} := e^{\overleftarrow{w}}(\overleftarrow{\mathcal{B}}(\mathcal{I}e^{-\overleftarrow{w}}|_{\partial SM}))_\psi$, and where $g_\pm \in \ker^0 \eta_\pm$, uniquely characterized by their boundary conditions

$$g_+|_{\partial M} = -i(\mathcal{I} - \vec{\mathcal{D}}|_{\partial SM})_0, \quad g_-|_{\partial M} = i(\mathcal{I} - \overleftarrow{\mathcal{D}}|_{\partial SM})_0.$$

Proof of Theorem 7.9. Let $e^{-\vec{w}}$ and $e^{-\overleftarrow{w}}$ holomorphic and antiholomorphic integrating factors for a (in particular, \vec{w} is an odd, holomorphic solution of $X\vec{w} = -a$ and \overleftarrow{w} is an odd, antiholomorphic solution of $X\overleftarrow{w} = -a$), so as to obtain

$$X(ue^{-\vec{w}}) = -(f + X_\perp h_0)e^{-\vec{w}} = -b(x, v),$$

where b is of the form $b_{-1} + b_0 + \mathcal{O}_{\geq 1}$ with, in particular, $b_{-1} = -\frac{1}{i}\eta_- h_0$. Thanks to Theorem 7.5, defining $v := ue^{-\vec{w}}$, the function $\vec{v} = v - (\vec{\mathcal{B}}(v|_{\partial SM}))_\psi$ is holomorphic and satisfies

$$X\vec{v} = -b.$$

Then defining $\vec{u} := e^{\vec{w}}\vec{v} = u - e^{\vec{w}}(\vec{\mathcal{B}}(ue^{-\vec{w}}|_{\partial SM}))_\psi = u - \vec{\mathcal{D}}$, \vec{u} solves the equation

$$X\vec{u} + a\vec{u} = -f - X_\perp h_0. \quad (35)$$

Similarly using the antiholomorphic integrating factor, the function $\overleftarrow{u} = u - e^{\overleftarrow{w}}(\overleftarrow{\mathcal{B}}(ue^{-\overleftarrow{w}}|_{\partial SM}))_\psi = u - \overleftarrow{\mathcal{D}}$ is antiholomorphic and solves

$$X\overleftarrow{u} + a\overleftarrow{u} = -f - X_\perp h_0. \quad (36)$$

Projecting (35) onto H_{-1} and (36) onto H_1 , we obtain

$$\begin{aligned} \eta_- \vec{u}_0 &= \frac{1}{i}\eta_- h_0 & \Leftrightarrow & \quad \eta_- (h_0 - i\vec{u}_0) = 0, \\ \eta_+ \overleftarrow{u}_0 &= -\frac{1}{i}\eta_+ h_0 & \Leftrightarrow & \quad \eta_+ (h_0 + i\overleftarrow{u}_0) = 0. \end{aligned}$$

This implies the relations:

$$\begin{aligned} h_0 - i\vec{u}_0 &= g_+ \in \ker^0 \eta_+, \\ h_0 + i\overleftarrow{u}_0 &= g_- \in \ker^0 \eta_-. \end{aligned} \tag{37}$$

which, since h_0 vanishes at the boundary, completely determines g_{\pm} from their boundary values, which are in turn determined from the boundary values of \vec{u} and \overleftarrow{u} , in turn determined by the data. Taking the half-sum, we obtain

$$h_0 = \frac{1}{2}(g_+ + g_-) - \frac{i}{2}(\vec{u}_0 - \overleftarrow{u}_0) = \frac{1}{2}(g_+ + g_-) - \frac{i}{2}(\vec{\mathcal{D}}_0 - \overleftarrow{\mathcal{D}}_0),$$

where the right-hand side is completely determined by data. On to the determination of f , we project the equation $Xu + au = -f - X_{\perp}h_0$ onto H_0 to make appear

$$f = -\eta_+ u_{-1} - \eta_- u_1 - au_0,$$

and show how to determine each term from the data. Since \vec{u} is holomorphic, then $\vec{u}_{-1} = 0 = u_{-1} - \vec{\mathcal{D}}_{-1}$, so $u_{-1} = \vec{\mathcal{D}}_{-1}$. Since \overleftarrow{u} is antiholomorphic, $\overleftarrow{u}_1 = 0 = u_1 - \overleftarrow{\mathcal{D}}_1$, so $u_1 = \overleftarrow{\mathcal{D}}_1$. Finally,

$$u_0 = \frac{1}{2}(\vec{u}_0 + \overleftarrow{u}_0 + \vec{\mathcal{D}}_0 + \overleftarrow{\mathcal{D}}_0) \stackrel{(37)}{=} \frac{1}{2}(\vec{\mathcal{D}}_0 + \overleftarrow{\mathcal{D}}_0) + \frac{i}{2}(g_+ - g_-).$$

We arrive at the following formula for f

$$f = -\eta_+ \vec{\mathcal{D}}_{-1} - \eta_- \overleftarrow{\mathcal{D}}_1 - \frac{a}{2} \left(\vec{\mathcal{D}}_0 + \overleftarrow{\mathcal{D}}_0 + i(g_+ - g_-) \right).$$

Theorem 7.9 is proved. \square

Theorem 7.9 gives rise to two linear operators

$$\begin{aligned} L_{a,0} &: \mathcal{I}_a^{0,\perp}(C_0^\infty(M) \times C^\infty(M)) \rightarrow C^\infty(M), \\ L_{a,\perp} &: \mathcal{I}_a^{0,\perp}(C_0^\infty(M) \times C^\infty(M)) \rightarrow C_0^\infty(M), \end{aligned}$$

such that

$$I_a L_{a,0}(I_a^0 f + I_a^\perp h_0) = I_a^0 f, \quad \text{and} \quad I_a L_{a,\perp}(I_a^0 f + I_a^\perp h_0) = I_a^\perp h_0.$$

If we then define $P_{a,0}, P_{a,\perp} : \text{Range } \mathcal{I}_a^{0,\perp} \rightarrow \text{Range } \mathcal{I}_a^{0,\perp}$, by

$$P_{a,0} := I_a L_{a,0}(Id - P_1 - P_{-1}), \quad P_{a,\perp} := I_a L_{a,\perp}(Id - P_1 - P_{-1}), \tag{38}$$

such operators are idempotent, projections of $\text{Range } \mathcal{I}_a$ onto $I_a^0(C^\infty(M))$ and $I_a^\perp(C_0^\infty(M))$, respectively.

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DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115, USA
E-mail address: `y_assylbekov@yahoo.com`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA CRUZ, CA 95064, USA
E-mail address: `fmonard@ucsc.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195-4350, USA. DEPARTMENT OF MATHEMATICS AND STATISTICS UNIVERSITY OF HELSINKI, BOX 68, HELSINKI, 00014, FINLAND. INSTITUTE FOR ADVANCED STUDY, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, HONG KONG SAR
E-mail address: `gunther@math.washington.edu`